FY3403 Particle physics
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Problem 14. The ground state of hydrogenlike atoms

The radial equation for the wave function of a hydrogenlike atom (i.e. a system with only one electron around the atomic nucleus) reads

$$\left[\frac{\hbar^2}{2m}\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\ell(\ell+1)}{r^2}\right)-\frac{Z\,e^2}{4\pi\varepsilon_0\,r}\right]R(r)=E\,R(r) \tag{1}$$

where Z is the nuclear charge (in units of the positron charge), and m is the reduced mass.

a) Assume $\ell = 0$, and show that the ground state ansatz $R(r) = \mathrm{e}^{-r/r_0}$ solves equation (1) for suitable choices of r_0 and E. Why must this be the ground state wave function?

b) Write down explicit expressions for r_0 and E. Express the answers in terms of the fine structure constant $\alpha = e^2/(4\pi\varepsilon_0\hbar c)$, the nuclear charge Z, the mass ratio m_e/M (where M is the mass of the nucleus), and (for r_0) the electron Compton wave length $\lambda_e = h/(m_e c)$ or (for E) the electron rest energy $m_e c^2$. Check that your answer agrees with the result for positronium, mentioned in the textbook chapter 5.3 (5.6 in the first edition), when you put $M = m_e$.

c) Calculate $\left\langle \left(\boldsymbol{p}^2\right)^2 \right\rangle$ in the ground state above. Compare your answer with equation (5.19) in the textbook, second edition ((5.42) in the first edition). Note that we neglect here the electron spin.
Hint: You may find the expression for $\boldsymbol{p}^2$ from equation (1).
Given: $\int_0^\infty \mathrm{d}r\, r^n\, \mathrm{e}^{-\beta r} = n!\,\beta^{-(n+1)}$, for n a nonnegative integer.

Problem 15. Relativistic splitting of the $n = 2$ level in positronium

As discussed in chapter 5.3 of the textbook, second edition (5.6 in the first edition), the $n = 2$ level in positronium is split into six different levels. Use the relevant formulas in this chapter to find the numerical values of

a) $E_{2\,{}^3S_1} - E_{2\,{}^1S_0}$,

b) $E_{2\,{}^1P_1} - E_{2\,{}^1S_0}$,

c) $E_{2\,{}^3P_0} - E_{2\,{}^1S_0}$,

d) $E_{2\,{}^3P_1} - E_{2\,{}^1S_0}$,

e) $E_{2\,{}^3P_2} - E_{2\,{}^1S_0}$.

Express the answers in eV.