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Solution to the exam in
FY3403 PARTICLE PHYSICS

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This solution consists of 5 pages.

Problem 1.

A particle of mass M decays into three particles of masses m_1, m_2, m_3. Let P, p_1, p_2, p_3 be the four-momenta of the decaying particle and of the three particles in the final state. Define the invariant mass m_{12} by the formula $m_{12}^2\,c^2 = (p_1+p_2)^2 = (P-p_3)^2$, and define m_{13} and m_{23} similarly.

Derive a relation between m_{12}^2 and the energy E_3 of particle 3 in the rest frame of the decaying particle.

$$m_{12}^2\,c^2 = (P-p_3)^2 = P^2 + p_3^2 - 2P\cdot p_3 = M^2c^2 + m_3^2c^2 - 2ME_3\ ,$$

since we have the 4-vectors $P = (Mc, 0, 0, 0)$ and $p_3 = (E_3/c, p_{3x}, p_{3y}, p_{3z})$.

Show that the sum $m_{12}^2 + m_{13}^2 + m_{23}^2$ is constant.

In the rest frame of the decaying particle we have that

$$(m_{12}^2+m_{13}^2+m_{23}^2)c^2 = (3M^2+m_3^2+m_2^2+m_1^2)c^2 - 2M(E_3+E_2+E_1) = (M^2+m_1^2+m_2^2+m_3^2)c^2\ ,$$

since $E_1+E_2+E_3 = Mc^2$. Thus, $m_{12}^2 + m_{13}^2 + m_{23}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$.

Or we calculate in the following way, using an arbitrary reference frame,

$$(m_{12}^2+m_{13}^2+m_{23}^2)c^2 = (3M^2+m_3^2+m_2^2+m_1^2)c^2 - 2P\cdot(p_3+p_2+p_1) = (M^2+m_1^2+m_2^2+m_3^2)c^2\ ,$$

since $p_1+p_2+p_3 = P$ and $P\cdot P = P^2 = M^2c^2$.

Problem 2.

The meson $\phi(1020)$ decays mainly into two K mesons, even though there is very little energy left over as kinetic energy in the final state. See the Meson Summary Table in the Particle Physics Booklet.
In the quark model the quark content of ϕ is $s\overline{s}$, and its decay mainly into K mesons is "explained" by the so called OZI rule (OZI for Okubo, Zweig and Iizuka).
The same rule is used to explain why the particle $J/\psi(3097)$, with quark content $c\overline{c}$, is exceptionally stable.

What is the OZI rule?

The OZI rule states that if a QCD Feynman diagram for some (decay) process may be cut in two by a line cutting gluon lines but no quark lines (and not separating quarks bound together in a particle), then that process is suppressed.

The OZI rule is even more efficient for the $\psi(3097)$ than for the $\phi(1020)$, because the decay modes that it does not suppress, $\psi \to D^+D^-$ and $\psi \to D^0\overline{D}^0$, are forbidden by energy conservation.

The next most frequent decay mode of ϕ is listed in the table as $\rho\pi + \pi^+\pi^-\pi^0$. The reason is clearly that it is difficult to distinguish between these final states. Why is it difficult?

How would you try to distinguish between the final states $\rho\pi$ and $\pi^+\pi^-\pi^0$?

The ρ meson is very short lived (10^{-23} s), and decays immediately into two π mesons. Hence, we do not observe directly the ρ meson, we observe only the two π mesons from its decay.
If we see the final state $\pi^+\pi^-\pi^0$, then we may try to decide whether or not it comes from the intermediate state $\rho^0\pi^0$ by measuring the invariant mass of the particle pair $\pi^+\pi^-$. The mass of the ρ meson is 776 ± 75 MeV/c^2.
Similarly, the invariant mass of the particle pair $\pi^+\pi^0$, or of the particle pair $\pi^-\pi^0$, would tell us whether the three π mesons come from the intermediate state $\rho^+\pi^-$, or from $\rho^-\pi^+$.
In practice we would make what is called a *Dalitz plot*: when we observe the decay process $\omega \to \pi^+\pi^-\pi^0$ we compute the invariant mass squared m^2_{+-} of the pair $\pi^+\pi^-$ and the invariant mass squared m^2_{+0} of the pair $\pi^+\pi^0$, and we plot a point in a two dimensional scatter plot with m^2_{+-} and m^2_{+0} on the two coordiate axes. The invariant mass squared m^2_{-0} of the pair $\pi^-\pi^0$ may be read off from the same plot as $m^2_{-0} = m^2_\omega + 3m^2_\pi - m^2_{+-} - m^2_{+0}$.
Much can be learned from the study of a Dalitz plot, about spin and parity of the three particle final state, and about two particle resonances like the $\rho \to 2\pi$ in this case, but that story is too long to be told here.

Problem 3.

Consider the following charge states of two π mesons: $|\,\pi^+\pi^-\rangle$, $|\,\pi^-\pi^+\rangle$ and $|\,\pi^0\pi^0\rangle$.

Use the table of Clebsch–Gordan coefficients, pages 273 and 274 in the Particle Physics Booklet, and write the decomposition of these states into states of different total isospin.

$$
\begin{aligned}
|\,\pi^+\,\pi^-\rangle &= \sqrt{\frac{1}{6}}\,|\,2,0\rangle + \sqrt{\frac{1}{2}}\,|\,1,0\rangle + \sqrt{\frac{1}{3}}\,|\,0,0\rangle\ , \\
|\,\pi^-\pi^+\,\rangle &= \sqrt{\frac{1}{6}}\,|\,2,0\rangle - \sqrt{\frac{1}{2}}\,|\,1,0\rangle + \sqrt{\frac{1}{3}}\,|\,0,0\rangle\ , \\
|\,\pi^0\pi^0\,\rangle &= \sqrt{\frac{2}{3}}\,|\,2,0\rangle - \sqrt{\frac{1}{3}}\,|\,0,0\rangle\ .
\end{aligned}
$$

We may observe that the state with total isospin $I = 1$ is antisymmetric in isospin under the interchange of the two particles (one consequence of the antisymmetry is that $I = 1$ does not contribute to the $\pi^0\pi^0$ state, which is symmetric in isospin). The $I = 2$ and $I = 0$ states are symmetric in isospin.

Problem 4.

The meson $\omega(782)$ has quantum numbers $I^G(J^{PC}) = 0^-(1^{--})$, see the Meson Summary Table.

$\omega(782)$ and $\phi(1020)$ are both called vector mesons. Why?

Because their spin and parity J^P is the spin and parity of a vector, 1^-.

$\omega(782)$ has three decay modes with branching ratios higher than 1 %.

Which interaction is responsible for each of these three decay modes?

Explain your reasoning (even if you have to guess the answers).

The three decay modes are $\omega \to \pi^+\pi^-\pi^0$ (branching ratio 89.2 %), $\omega \to \pi^0\gamma$ (8.92 %) and $\omega \to \pi^+\pi^-$ (1.53 %).

A good guess is that the first mode is due to the strong interaction. This is consistent with the conservation of G-parity, which is -1 for the ω meson and $(-1)^n$ for a state of n π mesons. Other conservation laws are not quite as easy to check, we have to assume that they hold. Just the fact that a three-body decay mode dominates over two-body decay modes indicates that the three-body mode is strong and the two-body modes are not strong.

The second mode is obviously electromagnetic, since a photon is involved.

The third mode is also most likely electromagnetic. G-parity is not conserved in the decay, it is $(-1)^2 = +1$ for two π mesons. This excludes the strong interaction, which conserves G-parity, but does not exclude the electromagnetic interaction.

We may exclude the weak interaction for all tree modes, simply because it is too weak. It would typically give a life-time about ten orders of magnitude larger.

The decay mode $\omega \to 3\pi^0$ is not observed, and an upper limit for the branching ratio is listed, under the heading "Charge conjugation (C) violating modes".

Explain why (how) it violates charge conjugation symmetry.

The π^0 is an eigenstate of C with eigenvalue $+1$. Hence a state of any number of π^0's, and no other particles, has $C = +1$.
Since ω has $C = -1$, it cannot decay into a number of π^0's, and nothing more, without violating C conservation.
We believe that C is a good symmetry for both strong and electromagnetic interactions.

The decay mode $\omega \to 2\pi^0$ is also not observed. It violates charge conjugation symmetry just as much as $\omega \to 3\pi^0$, but is not listed under "C violating modes".
In fact, $\omega \to 2\pi^0$ is not listed at all in the table. The reason may be that it is forbidden by a fundamental conservation law (more fundamental than charge conjugation symmetry), together with the experimental result that π mesons are bosons. Explain!

You may want to reconsider this problem after answering the next problems.

The π^0 is a boson, hence the wave function of a system with two π^0 must be symmetric under interchange of the two particles,

$$\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1) \ .$$

The symmetry of the wave function is $(-1)^{\ell_{\rm rel}}$, hence the relative orbital angular momentum of the $\pi^0\pi^0$ system must be even, $\ell_{\rm rel} = 0, 2, 4, \ldots$. This would be impossible in the decay $\omega \to \pi^0\pi^0$, unless we break the conservation law for angular momentum, which is among the most well established conservation laws.

Problem 5.

Let us consider the decay modes $\omega(782) \to 2\pi$, either $\pi^+\pi^-$ or $\pi^0\pi^0$.
We use always the rest frame of the decaying ω meson.
For the sake of the argument we choose to ignore the fact that $\pi^0\pi^0$ is forbidden.
What is the orbital angular momentum of the final state ($\ell_{\rm rel}$ in the notation introduced above)?

We always assume that angular momentum is conserved.

The π mesons in the final state have no spin, hence the total angular momentum in the final state, in the centre of mass reference system, is $\ell_{\rm rel} = 1$, assuming conservation of angular momentum.

It is possible to analyze the 2π final state without introducing the concept of isospin. Then we treat the π^+ and the π^- as distinguishable particles. The two π^0 mesons are identical bosons, and their wave function has to be symmetric, $\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$, but for the sake of the argument we will ignore this symmetry requirement here.
What is the parity P of the final state?
Is the parity different for the final states $\pi^+\pi^-$ and $\pi^0\pi^0$?

Is P conserved in the decay?

For a wave function of the product form $\psi(\vec{r}_1, \vec{r}_2) = \psi_{\rm CM}(\vec{R})\, \psi_{\rm rel}(\vec{r})$, with

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \ , \qquad \vec{r} = \vec{r}_1 - \vec{r}_2 \ ,$$

the parity transformation $\vec{r}_1 \to -\vec{r}_1$ and $\vec{r}_2 \to -\vec{r}_2$ gives $\vec{R} \to -\vec{R}$ and $\vec{r} \to -\vec{r}$. Hence,

$$\psi(-\vec{r}_1, -\vec{r}_2) = \psi_{\rm CM}(-\vec{R})\, \psi_{\rm rel}(-\vec{r}) = (-1)^{\ell_{\rm rel}}\, \psi(\vec{r}_1, \vec{r}_2) \ .$$

The internal parity of two π mesons is $(-1)^2 = +1$, hence the parity of the final state is $(-1)^{\ell_{\rm rel}} = (-1)^1 = -1$.
Conclusion: Parity is conserved in the decay.
The analysis is the same for the $\pi^+\pi^-$ and the $\pi^0\pi^0$ final states, when we ignore the impossibility of having two π^0's in an antisymmetric state.

What is the charge conjugation symmetry C of the final state?
Is the charge conjugation symmetry different for the final states $\pi^+\pi^-$ and $\pi^0\pi^0$?

Is C conserved in the decay?

In the case of the $\pi^+\pi^-$ state, the charge conjugation is the same as the interchange $\vec{r}_1 \leftrightarrow \vec{r}_2$, which gives $\vec{R} \leftrightarrow \vec{R}$ and $\vec{r} \leftrightarrow -\vec{r}$. Hence we have $C = (-1)^{\ell_{\rm rel}} = -1$, and C is conserved in the decay.

The $\pi^0\pi^0$ state is an eigenstate of charge conjugation, with eigenvalue $(+1)^2 = +1$, independent of the spatial wave function. In this case C is not conserved in the decay.

Problem 6.

We consider still the decay modes $\omega(782) \to 2\pi$.
Now we introduce isospin, and we write the total wave function of the two π mesons as a product

$$\Psi(Q_1, \vec{r}_1; Q_2, \vec{r}_2) = \psi_{\text{isospin}}(Q_1, Q_2)\, \psi_{\text{space}}(\vec{r}_1, \vec{r}_2) \ .$$

Here $Q = I_3$ is the electric charge, I_3 being the third component of the isospin.
Then we treat the two π mesons as identical bosons, even if they have different charges, and we require the total wave function to be symmetric under interchange,

$$\Psi(Q_1, \vec{r}_1; Q_2, \vec{r}_2) = \Psi(Q_2, \vec{r}_2; Q_1, \vec{r}_1) \ .$$

The symmetry restriction on the total wave function restricts the total isospin in the final state.

What is the restriction on the total isospin?

Is the isospin conserved in the decay?

We have seen that the spatial part of the wave function must be antisymmetric under interchange, because it must have relative orbital angular momentum $\ell_{\text{rel}} = 1$.
Hence, in order to make the total wave function symmetric, we need an antisymmetric isospin wave function. This requires total isospin $I = 1$, as we saw in Problem 3 above.
Since the ω has isospin 0, isospin is not conserved in the decay. This conclusion holds for both the final states $\pi^+\pi^-$ and $\pi^0\pi^0$.

It is worth noting that we arrive at the conclusion that $I = 1$ in the final state by postulating that the π^+ and the π^- are identical bosons and must have a symmetric total wave function, when we include the isospin part of the wave function.

We might be tempted to claim that π^+ and π^- are not identical particles, because the value of the charge disinguishes a π^+ from a π^-. But in the same sense an electron on the Moon is not identical to an electron on the Earth, because we may distinguish them simply by observing their positions. And nobody would claim that two electrons are not identical?

There is no difference in principle between these two ways of distinguishing particles, either by measuring electrical charge, or by measuring position.