Chapter 7

Classification of Lie algebras

The present chapter is only a rough sketch of the theory behind the classification of Lie algebras.

An *invariant subalgebra*, or *ideal*, of a Lie algebra $\mathcal{L}$ is a subalgebra $\mathcal{M} \subset \mathcal{L}$ such that $[L, M] \in \mathcal{M}$ for all $L \in \mathcal{L}$ and $M \in \mathcal{M}$. The two trivial examples are $\mathcal{M} = \mathcal{L}$ and $\mathcal{M} = \{0\}$.

A *simple* Lie algebra is non-Abelian (not all commutators vanish) and has no non-trivial invariant subalgebra.

A *semisimple* Lie algebra has no Abelian invariant subalgebra.

If $\mathcal{L}$ is semisimple, then it is non-Abelian, since it is a subalgebra of itself, and it is a direct sum of one or more simple Lie algebras. This means that $\mathcal{L}$ as a vector space is a direct sum of vector spaces,

$$\mathcal{L} = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_k \ , \tag{7.1}$$

where each $\mathcal{S}_i$ is a simple Lie algebra, and $[L, M] = 0$ if $L \in \mathcal{S}_i$ and $M \in \mathcal{S}_j$ with $i \neq j$.

Every Lie algebra $\mathcal{L}$, either real or complex, which is not semisimple, is an *indirect sum* of a *solvable* Lie algebra $\mathcal{N}$ and a semisimple Lie algebra $\mathcal{S}$. This means that $\mathcal{L}$ as a vector space is the direct sum of vector spaces,

$$\mathcal{L} = \mathcal{N} \oplus \mathcal{S} \ , \tag{7.2}$$

such that $[S, N] \in \mathcal{N}$ when $S \in \mathcal{S}$ and $N \in \mathcal{N}$.

That $\mathcal{N}$ is solvable means that it contains a sequence of subalgebras $\mathcal{N}^{(i)}$ with $\mathcal{N}^{(0)} = \mathcal{N}$ and

$$\mathcal{N}^{(i+1)} = [\,\mathcal{N}^{(i)}, \mathcal{N}^{(i)}\,] = \{\,[L, M] \mid L, M \in \mathcal{N}^{(i)}\} \ , \tag{7.3}$$

terminating for some $l \geq 0$ with $\mathcal{N}^{(l+1)} = \{0\}$. Thus, $\mathcal{N}^{(l)}$ is an Abelian subalgebra of $\mathcal{L}$, which proves that $\mathcal{L}$ is not semisimple.

This classification theorem means that the classification of the simple Lie algebras is a central part of the classification of all Lie algebras, in the same way as the classification of the simple finite groups is a central part of the classification of all finite groups.

7.1 Scalar product

There is a natural scalar product on any real or complex Lie algebra $\mathcal{L}$, called the *Killing form*, or *Cartan metric*, defined as

$$(L, M) = \mathrm{Tr}(\mathrm{Ad}(L)\,\mathrm{Ad}(M)) \ . \tag{7.4}$$

This scalar product is symmetric, $(M, L) = (L, M)$. It is not Hermitean when $\mathcal{L}$ is a complex Lie algebra, since $(M, L) \neq (L, M)^*$ when (L, M) is complex. Recall the definition that $\mathrm{Ad}(L)$ for $L \in \mathcal{L}$ is a linear operator acting on $\mathcal{L}$ by commutation,

$$\mathrm{Ad}(L)\, M = [L, M] \ . \tag{7.5}$$

Given a set of basis vectors $E_1, E_2, \ldots, E_n$, with commutation relations

$$[E_i, E_j] = \sum_k f_{ij}^k E_k \ , \tag{7.6}$$

the scalar product is defined by the metric tensor

$$g_{ij} = g_{ji} = (E_i, E_j) = \mathrm{Tr}(\mathrm{Ad}(E_i)\, \mathrm{Ad}(E_j)) = \sum_k \sum_l f_{il}^k f_{jk}^l \ . \tag{7.7}$$

This Cartan metric tensor provides a very simple criterion for semisimplicity. The Lie algebra is semisimple if and only if g_{ij} is a non-singular matrix.

We will assume from now on that our Lie algebra $\mathcal{L}$ is semisimple, so that the scalar product is non-singular. For every linear operator A on $\mathcal{L}$ we define its transpose $A^\top$ by the condition that

$$(L, A^\top M) = (AL, M) \qquad \forall\, L, M \in \mathcal{L} \ . \tag{7.8}$$

We say that A is symmetric if $A^\top = A$, antisymmetric if $A^\top = -A$, and orthogonal if $A^\top = A^{-1}$. With these definitions, $\mathrm{Ad}(L)$ is antisymmetric, and the exponential $\exp(\mathrm{Ad}(L))$ is an orthogonal transformation on $\mathcal{L}$, for every $L \in \mathcal{L}$.

The antisymmetry $\mathrm{Ad}(L)^\top = -\mathrm{Ad}(L)$ means by definition that

$$(\mathrm{Ad}(L)M, N) = -(M, \mathrm{Ad}(L)N) \tag{7.9}$$

for all $L, M, N \in \mathcal{L}$. It is proved in the following way. Since

$$\mathrm{Ad}([L, M]) = [\mathrm{Ad}(L), \mathrm{Ad}(M)] = \mathrm{Ad}(L)\, \mathrm{Ad}(M) - \mathrm{Ad}(M)\, \mathrm{Ad}(L) \ , \tag{7.10}$$

we have that

$$\begin{aligned} (\mathrm{Ad}(L)M, N) &= ([L, M], N) = \mathrm{Tr}(\mathrm{Ad}([L, M])\, \mathrm{Ad}(N)) \\ &= \mathrm{Tr}(\mathrm{Ad}(L)\, \mathrm{Ad}(M)\, \mathrm{Ad}(N)) - \mathrm{Tr}(\mathrm{Ad}(M)\, \mathrm{Ad}(L)\, \mathrm{Ad}(N)) \ , \end{aligned} \tag{7.11}$$

and similarly,

$$\begin{aligned} (M, \mathrm{Ad}(L)N) &= (\mathrm{Ad}(L)N, M) \\ &= \mathrm{Tr}(\mathrm{Ad}(L)\, \mathrm{Ad}(N)\, \mathrm{Ad}(M)) - \mathrm{Tr}(\mathrm{Ad}(N)\, \mathrm{Ad}(L)\, \mathrm{Ad}(M)) \\ &= \mathrm{Tr}(\mathrm{Ad}(M)\, \mathrm{Ad}(L)\, \mathrm{Ad}(N)) - \mathrm{Tr}(\mathrm{Ad}(L)\, \mathrm{Ad}(M)\, \mathrm{Ad}(N)) \ , \end{aligned} \tag{7.12}$$

where we have used the general relation $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$.

7.2 Projective group representations

It is convenient at this point to introduce the Dirac notation. A quantum state of a physical system is represented by a "ket" vector $|\psi\rangle$ in a Hilbert space. The scalar product of two ket vectors $|\psi\rangle$ and $|\phi\rangle$ is the "bracket" $\langle\psi|\phi\rangle$, where the "bra" vector

$$\langle\psi| = (|\psi\rangle)^\dagger \tag{7.13}$$

is the Hermitean conjugate of the ket vector $|\psi\rangle$. The physical interpretation is that

$$P = |\langle\psi|\phi\rangle|^2 \tag{7.14}$$

is the probability of observing the state $|\psi\rangle$ if the state is $|\phi\rangle$.

This probability interpretation requires the normalization

$$\langle\psi|\psi\rangle = 1 \tag{7.15}$$

for a ket vector $|\psi\rangle$ representing a physical state. Furthermore, it implies that the ket vectors $|\psi\rangle$ and $\mathrm{e}^{\mathrm{i}\alpha}|\psi\rangle$ represent the same physical state if α is an arbitrary real phase angle.

From the mathematical point of view it is the one dimensional subspace

$$\Psi = \{\, c\,|\psi\rangle \mid c \in \mathbb{C}\,\} \tag{7.16}$$

also called a *ray*, or a *line*, in the Hilbert space, that represents the physical state. If the Hilbert space is $\mathbb{C}^n$, then the space of all one dimensional subspaces is the *complex projective space* $\mathbb{CP}^{n-1}$.

A transformation of the physical system, for example a rotation R, transforms a state Ψ into $\widetilde{\Psi} = R(\Psi)$. A minimum requirement for R to be a symmetry is that it preserves probabilities. That is, if $|\psi\rangle$, $|\phi\rangle$, $|\widetilde{\psi}\rangle$, $|\widetilde{\phi}\rangle$ are normalized state vectors with $|\psi\rangle \in \Psi$, $|\phi\rangle \in \Phi$, $|\widetilde{\psi}\rangle \in \widetilde{\Psi} = R(\Psi)$, $|\widetilde{\phi}\rangle \in \widetilde{\Phi} = R(\Phi)$, then we should have

$$|\langle\widetilde{\psi}|\widetilde{\phi}\rangle| = |\langle\psi|\phi\rangle| \;. \tag{7.17}$$

It is a theorem by Wigner, with a somewhat lengthy proof, that any probability preserving transformation on the projective space $\mathbb{CP}^{n-1}$ can be represented as an invertible transformation U on $\mathbb{C}^n$ which is either linear,

$$U(a\,|\psi\rangle + b\,|\phi\rangle) = a\,|\widetilde{\psi}\rangle + b\,|\widetilde{\phi}\rangle \;, \tag{7.18}$$

with $|\widetilde{\psi}\rangle = U\,|\psi\rangle$, $|\widetilde{\phi}\rangle = U\,|\phi\rangle$, and unitary,

$$\langle\widetilde{\psi}|\widetilde{\phi}\rangle = \langle\psi|\phi\rangle \;, \tag{7.19}$$

or else is antilinear,

$$U(a\,|\psi\rangle + b\,|\phi\rangle) = a^*\,|\widetilde{\psi}\rangle + b^*\,|\widetilde{\phi}\rangle \;, \tag{7.20}$$

and antiunitary,

$$\langle\widetilde{\psi}|\widetilde{\phi}\rangle = \langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle \;. \tag{7.21}$$

In practice, the only example of an antiunitary symmetry is time reversal.

Obviously, if U is a unitary transformation representing one particular physical symmetry transformation, and if α is an arbitrary real phase angle, then $\mathrm{e}^{\mathrm{i}\alpha}U$ represents the same physical transformation.

Hence, if every rotation R is represented by a unitary transformation $U(R)$, this need not be a unitary representation of the rotation group SO(3). The most general possibility is a unitary representation up to phase factors, so that when $R_3 = R_2 R_1$ we have

$$U(R_2)\,U(R_1) = \mathrm{e}^{\mathrm{i}\gamma}\,U(R_3)\;, \tag{7.22}$$

with a real phase angle γ depending on R_1, R_2 and R_3. A representation up to phase factors of the rotation group, or any other group, is called a *projective representation*, because it represents the group as a group of transformations on the projective space.

It may be possible to choose the arbitrary phase factor which is implicit in every unitary transformation $U(R)$ so as to get $\gamma = 0$ in eq. (7.22). Then we have a unitary representation of SO(3). Again there is a non-trivial theorem saying that if we can not get a unitary representation of SO(3), then we can get instead a unitary representation of its universal covering group SU(2). A unitary representation of either SO(3) or SU(2) is anyway a linear representation of the Lie algebra so(3).

In conclusion, *the rotation group SO*(3) *acts on the projective space of quantum states in such a way that there is a linear representation of the Lie algebra so*(3) $\simeq$ *su*(2).

7.3 Unitary representations of SU(2)

The unitary representations of the unitary group SU(2) should be familiar from quantum mechanics. Here is a brief review of the construction, which can be generalized to other Lie algebras.

We represent the three su(2) generators $\boldsymbol{\mu}_i$, $i = 1, 2, 3$, as operators on the Hilbert space,

$$\boldsymbol{\mu}_i \to -\frac{\mathrm{i}}{\hbar}\,J_i\;. \tag{7.23}$$

The physical interpretation is that the three dimensional vector $\vec{J}$ is the angular momentum. Since the components of the angular momentum are physical observables, they should be Hermitean operators, $J_i^{\dagger} = J_i$. This condition ensures that the exponential

$$U = \exp\!\left(-\frac{\mathrm{i}}{\hbar}\,\alpha\,\vec{n}\cdot\vec{J}\right), \tag{7.24}$$

representing the rotation by an angle α about an axis $\vec{n}$, is a unitary transformation.

The basic commutation relations

$$[\boldsymbol{\mu}_1, \boldsymbol{\mu}_2] = \boldsymbol{\mu}_3\;, \quad [\boldsymbol{\mu}_2, \boldsymbol{\mu}_3] = \boldsymbol{\mu}_1\;, \quad [\boldsymbol{\mu}_3, \boldsymbol{\mu}_1] = \boldsymbol{\mu}_2\;, \tag{7.25}$$

lead to the angular momentum commutation relations

$$[J_1, J_2] = \mathrm{i}\hbar J_3\;, \quad [J_2, J_3] = \mathrm{i}\hbar J_1\;, \quad [J_3, J_1] = \mathrm{i}\hbar J_2\;. \tag{7.26}$$

By Schur's lemma, every linear operator commuting with all group transformations in an irreducible group representation is a multiple of the identity I.

Therefore it is useful to know *Casimir operators*, which commute with every member of the Lie algebra. A Casimir operator in a linear representation of a Lie algebra is usually not a member of the Lie algebra, but is typically a polynomial in the basic generators. In the case of su(2) there is one single Casimir operator, it is the length squared of the angular momentum vector,

$$\vec{J}^{\,2} = J_1^2 + J_2^2 + J_3^2 \; . \tag{7.27}$$

The Leibniz rules of commutation,

$$[A, BC] = [A, B]\, C + B\, [A, C] \; , \qquad [AB, C] = [A, C]\, B + A\, [B, C] \; , \tag{7.28}$$

are useful for computing commutators. They give for example that

$$\begin{aligned} [\vec{J}^{\,2}, J_1] &= [J_1, J_1]\, J_1 + J_1\, [J_1, J_1] + [J_2, J_1]\, J_2 + J_2\, [J_2, J_1] + [J_3, J_1]\, J_3 + J_3\, [J_3, J_1] \\ &= \mathrm{i}\hbar\, (0 + 0 - J_3 J_2 - J_2 J_3 + J_2 J_3 + J_3 J_2) = 0 \; . \end{aligned} \tag{7.29}$$

Similarly, $[\vec{J}^{\,2}, J_2] = [\vec{J}^{\,2}, J_3] = 0$.

We also define the operators

$$J_\pm = J_1 \pm \mathrm{i} J_2 \; . \tag{7.30}$$

It is straightforward to verify the commutation relations

$$[J_3, J_+] = \hbar J_+ \; , \qquad [J_3, J_-] = -\hbar J_- \; , \qquad [J_+, J_-] = 2\hbar J_3 \; , \tag{7.31}$$

and the relations

$$\vec{J}^{\,2} = J_+ J_- + J_3^2 - \hbar J_3 = J_- J_+ + J_3^2 + \hbar J_3 \; . \tag{7.32}$$

Note that J_+ and J_- are not generators of SU(2) transformations, because they contain the imaginary unit i. Strictly speaking, when we introduce these operators we go from the real Lie algebra su(2) into its complex extension, which is $\mathrm{sl}(2, \mathbb{C})$.

We now choose $\vec{J}^{\,2}$ and J_3 as a pair of commuting operators to be quantized simultaneously. We call the eigenvectors $|\lambda m\rangle$, and write the eigenvalue equations

$$\vec{J}^{\,2}\, |\lambda m\rangle = \lambda \hbar^2\, |\lambda m\rangle \; , \qquad J_3\, |\lambda m\rangle = m\hbar\, |\lambda m\rangle \; . \tag{7.33}$$

Now start with one particular eigenvector $|\lambda m\rangle$, normalized such that

$$\langle \lambda m | \lambda m \rangle = 1 \; . \tag{7.34}$$

Define

$$|\psi\rangle = J_+\, |\lambda m\rangle \; , \qquad |\phi\rangle = J_-\, |\lambda m\rangle \; . \tag{7.35}$$

Because $\vec{J}^{\,2}$ commutes with all J_i, $|\psi\rangle$ and $|\phi\rangle$ are still eigenvectors of $\vec{J}^{\,2}$ with eigenvalue $\lambda\hbar^2$, for example,

$$\vec{J}^{\,2}\, |\psi\rangle = \vec{J}^{\,2} J_+\, |\lambda m\rangle = J_+ \vec{J}^{\,2}\, |\lambda m\rangle = \lambda\hbar^2\, J_+\, |\lambda m\rangle = \lambda\hbar^2\, |\psi\rangle \; . \tag{7.36}$$

Similarly,

$$\begin{aligned} J_3 \left|\psi\right\rangle &= J_3 J_+ \left|\lambda m\right\rangle = (J_3 J_+ - J_+ J_3 + J_+ J_3) \left|\lambda m\right\rangle = (\hbar J_+ + J_+ J_3) \left|\lambda m\right\rangle \\ &= (m+1)\hbar J_+ \left|\lambda m\right\rangle = (m+1)\hbar \left|\psi\right\rangle \,, \end{aligned} \tag{7.37}$$

and

$$\begin{aligned} J_3 \left|\phi\right\rangle &= J_3 J_- \left|\lambda m\right\rangle = (J_3 J_- - J_- J_3 + J_- J_3) \left|\lambda m\right\rangle = (-\hbar J_- + J_- J_3) \left|\lambda m\right\rangle \\ &= (m-1)\hbar J_- \left|\lambda m\right\rangle = (m-1)\hbar \left|\phi\right\rangle \,. \end{aligned} \tag{7.38}$$

Note that if we act first with J_+ and afterwards with J_-, we get back the same eigenvector $\left|\lambda m\right\rangle$ we started with,

$$J_- J_+ \left|\lambda m\right\rangle = (\vec{J}^{\,2} - J_3^2 - \hbar J_3) \left|\lambda m\right\rangle = (\lambda - m^2 - m)\hbar^2 \left|\lambda m\right\rangle \,. \tag{7.39}$$

In the same way, if we act first with J_- and next with J_+, we get

$$J_+ J_- \left|\lambda m\right\rangle = (\vec{J}^{\,2} - J_3^2 + \hbar J_3) \left|\lambda m\right\rangle = (\lambda - m^2 + m)\hbar^2 \left|\lambda m\right\rangle \,. \tag{7.40}$$

Hence, we may use J_+ and J_- to climb up and down a ladder where the value of λ stays constant, while the values of m differ by one. If we climb up and down again, or down and up again, we always return to the same eigenvector we started from, thus we get one single eigenvector $\left|\lambda m\right\rangle$ for each value of m.

Since $J_+^\dagger = J_-$, the norm squared of $\left|\psi\right\rangle = J_+ \left|\lambda m\right\rangle$ is

$$\langle\psi|\psi\rangle = \langle\lambda m|J_- J_+|\lambda m\rangle = (\lambda - m^2 - m)\hbar^2 \,. \tag{7.41}$$

Similarly, the norm squared of $\left|\phi\right\rangle = J_- \left|\lambda m\right\rangle$ is

$$\langle\phi|\phi\rangle = \langle\lambda m|J_+ J_-|\lambda m\rangle = (\lambda - m^2 + m)\hbar^2 \,. \tag{7.42}$$

The norm squared is not allowed to be negative, hence we can not allow $|m|$ to become arbitrarily large. The only way to stop the climbing up and down by $J_\pm$ is that there exists a maximal $m = m_2$ and a minimal $m = m_1$, such that

$$J_+ \left|\lambda m_2\right\rangle = 0 \,, \qquad J_- \left|\lambda m_1\right\rangle = 0 \,. \tag{7.43}$$

It follows from the above that

$$\lambda = m_2^2 + m_2 = m_1^2 - m_1 \,. \tag{7.44}$$

The last equation may be written as

$$(m_2 + m_1)(m_2 - m_1 + 1) = 0 \,. \tag{7.45}$$

Since we may climb up from m_1 to m_2 in $k = 0, 1, 2, \ldots$ steps of one, we must have that

$$m_2 - m_1 = k = 0, 1, 2, \ldots \,, \qquad m_2 + m_1 = 0 \,. \tag{7.46}$$

The solutions are

$$m_2 = -m_1 = j = \frac{k}{2} = 0 \,, \, \frac{1}{2} \,, \, 1 \,, \, \frac{3}{2} \,, \, 2 \,, \, \ldots \,, \qquad \lambda = j(j+1) \,. \tag{7.47}$$

To summarize, the quantum number $j = 0, 1/2, 1, 3/2, 2, \ldots$ identifies an irreducible unitary representation of the group SU(2). We may define normalized basis vectors $|jm\rangle$ in the Hilbert space, with j constant and m varying in integer steps from $m = -j$ to $m = j$, such that

$$\begin{aligned} J_3 \, |jm\rangle &= m\hbar \, |jm\rangle \, , \\ J_\pm \, |jm\rangle &= \hbar \sqrt{j(j+1) - m(m \pm 1)} \; |j, m \pm 1\rangle \\ &= \hbar \sqrt{\left(j + \frac{1}{2}\right)^2 - \left(m \pm \frac{1}{2}\right)^2} \; |j, m \pm 1\rangle \, . \end{aligned} \tag{7.48}$$

This defines also the action of the angular momentum operators

$$J_1 = \frac{1}{2}\,(J_+ + J_-)\, , \qquad J_2 = -\frac{\mathrm{i}}{2}\,(J_+ - J_-)\, . \tag{7.49}$$

And it follows that

$$\vec{J}^{\,2} \, |j, m\rangle = (J_1^2 + J_2^2 + J_3^2) \, |j, m\rangle = j(j+1)\hbar^2 \, |j, m\rangle \, . \tag{7.50}$$

Recall that all rotations by a given rotation angle α about different rotation axes belong to the same conjugation class in SO(3) or in SU(2). Thus, we may compute the character of a rotation R with rotation angle α by computing the trace of the unitary transformation representing a rotation an angle α about the z axis,

$$\chi^{(j)}(R) = \chi^{(j)}(\alpha) = \mathrm{Tr}\, \exp\!\left(-\frac{\mathrm{i}}{\hbar}\,\alpha J_3\right) = \sum_{m=-j}^{j} \mathrm{e}^{-\mathrm{i}\alpha m} = \frac{\sin((j+\frac{1}{2})\alpha)}{\sin(\frac{\alpha}{2})}\, . \tag{7.51}$$

7.4 Root diagrams of simple complex Lie algebras

Take $\mathcal{L} = \mathrm{sl}(3, \mathbb{C})$, the Lie algebra of $\mathrm{SL}(3, \mathbb{C})$, as an example. Its defining representation consists of the traceless complex 3×3 matrices, and we may choose the following eight matrices as a basis,

$$\begin{aligned} \mathbf{E}_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathbf{E}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathbf{E}_4 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{E}_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{E}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ \mathbf{E}_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{E}_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{7.52}$$

The first two lines of the commutation table, in the form

$$\begin{array}{c|c} & \mathbf{M} \\ \hline \mathbf{L} & [\mathbf{L}, \mathbf{M}] \end{array} \tag{7.53}$$

look as follows,

$$
\begin{array}{c|cccccccc}
 & \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 & \mathbf{E}_4 & \mathbf{E}_5 & \mathbf{E}_6 & \mathbf{E}_7 & \mathbf{E}_8 \\
\hline
\mathbf{E}_1 & \mathbf{0} & \mathbf{0} & 2\mathbf{E}_3 & -2\mathbf{E}_4 & \mathbf{E}_5 & -\mathbf{E}_6 & -\mathbf{E}_7 & \mathbf{E}_8 \\
\mathbf{E}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 3\mathbf{E}_5 & -3\mathbf{E}_6 & 3\mathbf{E}_7 & -3\mathbf{E}_8
\end{array}
\tag{7.54}
$$

We see that all eight basis vectors $\mathbf{E}_i \in \mathcal{L}$ are common eigenvectors for $\mathrm{Ad}(\mathbf{E}_1)$ and $\mathrm{Ad}(\mathbf{E}_2)$. It is possible for these two operators to have a complete set of common eigenvectors because they commute,

$$[\mathrm{Ad}(\mathbf{E}_1), \mathrm{Ad}(\mathbf{E}_2)] = \mathrm{Ad}([\mathbf{E}_1, \mathbf{E}_2]) = 0 \ . \tag{7.55}$$

The Lie algebra $\mathcal{L} = \mathrm{sl}(3, \mathbb{C})$ as a complex vector space is a direct sum

$$\mathcal{L} = \mathcal{H} \oplus \mathcal{K} \ , \tag{7.56}$$

where $\mathbf{E}_1, \mathbf{E}_2$ are basis vectors of $\mathcal{H}$ and $\mathbf{E}_3, \mathbf{E}_4, \ldots, \mathbf{E}_8$ are basis vectors of $\mathcal{K}$. $\mathcal{H}$ is an Abelian subalgebra of $\mathcal{L}$, that is, $[\mathbf{L}, \mathbf{M}] = \mathbf{0}$ for all $\mathbf{L}, \mathbf{M} \in \mathcal{H}$, and it is maximal in this respect: it is not contained in a larger Abelian subalgebra. Furthermore, we see that $[\mathbf{L}, \mathbf{M}] \in \mathcal{K}$ if $\mathbf{L} \in \mathcal{H}$ and $\mathbf{M} \in \mathcal{K}$. A subalgebra with such properties is called a *Cartan subalgebra*. There are of course infinitely many ways to choose a Cartan subalgebra of a simple Lie algebra, but all Cartan subalgebras are equivalent for classification purposes.

The *rank* of a simple Lie algebra is the dimension of a Cartan subalgebra. Thus, $\mathrm{sl}(3, \mathbb{C})$ has rank two. More generally, the rank of $\mathrm{sl}(n, \mathbb{C})$ is $n-1$, since this is the number of linearly independent traceless diagonal matrices.

From the commutation table in eq. (7.54) we may read off some components of the Cartan metric tensor

$$g_{ij} = (\mathbf{E}_i, \mathbf{E}_j) = \mathrm{Tr}(\mathrm{Ad}(\mathbf{E}_i)\,\mathrm{Ad}(\mathbf{E}_j)) \ . \tag{7.57}$$

We get

$$
\begin{aligned}
g_{11} &= 2^2 + (-2)^2 + 1^2 + (-1)^2 + (-1)^2 + 1^2 = 12 \ , \\
g_{22} &= 3^2 + (-3)^2 + 3^2 + (-3)^2 = 36 \ , \\
g_{12} &= 1 \cdot 3 + (-1) \cdot (-3) + (-1) \cdot 3 + 1 \cdot (-3) = 0 \ .
\end{aligned}
\tag{7.58}
$$

Thus, the basis vectors $\mathbf{E}_1, \mathbf{E}_2$ of the Cartan subalgebra $\mathcal{H}$ are orthogonal, but are not normalized. Normalized orthogonal basis vectors of the Cartan subalgebra are

$$\mathbf{H}_1 = \frac{1}{2\sqrt{3}}\,\mathbf{E}_1 \ , \qquad \mathbf{H}_2 = \frac{1}{6}\,\mathbf{E}_2 \ . \tag{7.59}$$

Let $\mathbf{E} \in \mathcal{L}$ be a common eigenvector of $\mathrm{Ad}(\mathbf{H}_1)$ and $\mathrm{Ad}(\mathbf{H}_2)$,

$$[\mathbf{H}_1, \mathbf{E}] = \alpha_1 \mathbf{E} \ , \qquad [\mathbf{H}_2, \mathbf{E}] = \alpha_2 \mathbf{E} \ . \tag{7.60}$$

Because the rank of the Lie algebra is two, there are two basis vectors $\mathbf{H}_1, \mathbf{H}_2$ of the Cartan subalgebra and two eigenvalues α_1 and α_2. We introduce a two dimensional *root vector*, or root, $\alpha = (\alpha_1, \alpha_2)$. Apart from the root vector $\alpha = (0, 0)$ corresponding to the two eigenvectors $\mathbf{E} = \mathbf{H}_1, \mathbf{H}_2$, there are six different nonzero root vectors corresponding to the six eigenvectors $\mathbf{E} = \mathbf{E}_3, \mathbf{E}_4, \ldots, \mathbf{E}_8$, as summarized in the following table.

	$\mathbf{E}_3$	$\mathbf{E}_4$	$\mathbf{E}_5$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
α_1	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{2\sqrt{3}}$
α_2	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

(7.61)

Plotting all eight root vectors we have the picture of $\mathrm{sl}(3, \mathbb{C})$ shown in Figure 7.1.

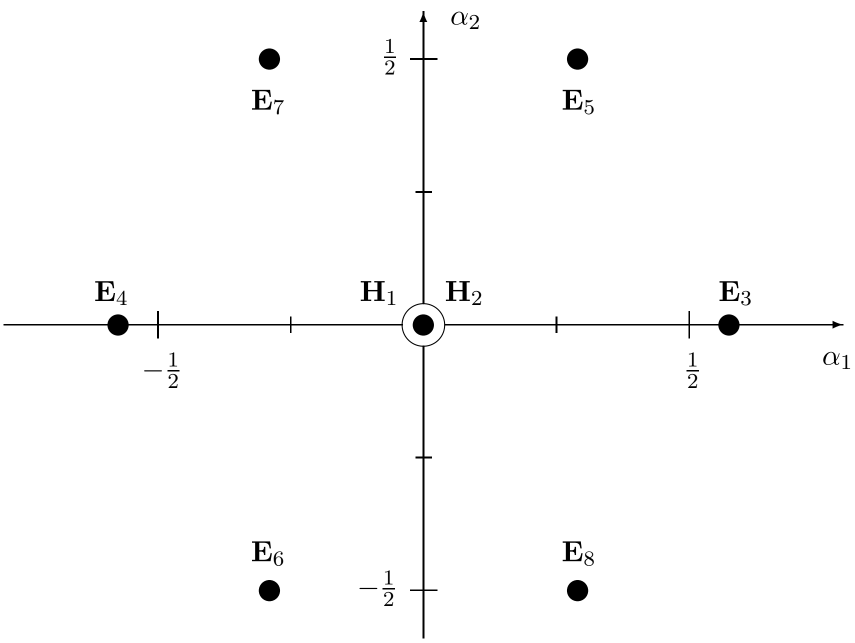

Figure 7.1: Root diagram of $\mathrm{sl}(3, \mathbb{C})$.

If $\mathbf{E}_\alpha$ and $\mathbf{E}_\beta$ are eigenvectors corresponding to the roots $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, then either they commute, or else their commutator is an eigenvector corresponding to the sum of the roots, $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$. This follows from the Jacobi identity, which gives that

$$\begin{aligned} [\mathbf{H}_i, [\mathbf{E}_\alpha, \mathbf{E}_\beta]] &= [[\mathbf{H}_i, \mathbf{E}_\alpha], \mathbf{E}_\beta] + [\mathbf{E}_\alpha, [\mathbf{H}_i, \mathbf{E}_\beta]] \\ &= [\alpha_i \mathbf{E}_\alpha, \mathbf{E}_\beta] + [\mathbf{E}_\alpha, \beta_i \mathbf{E}_\beta] = (\alpha_i + \beta_i)\, [\mathbf{E}_\alpha, \mathbf{E}_\beta] \; . \end{aligned} \tag{7.62}$$

For example, we see from Figure 7.1 that $\mathbf{E}_3$ and $\mathbf{E}_5$ have roots α and β such that $\alpha + \beta$ is not a root. It follows that

$$[\mathbf{E}_3, \mathbf{E}_5] = 0 \; . \tag{7.63}$$

On the other hand, $\mathbf{E}_3$ and $\mathbf{E}_7$ have roots α and β such that $\alpha + \beta$ is the root of $\mathbf{E}_5$, hence

$$[\mathbf{E}_3, \mathbf{E}_7] = f_{37}^{5}\, \mathbf{E}_5 \; , \tag{7.64}$$

with a structure constant f_{37}^5 which is so far undetermined.

Next, $\mathbf{E}_3$ and $\mathbf{E}_4$ have roots α and $-\alpha$ such that $\alpha + (-\alpha) = (0,0)$. It follows that their commutator belongs to the Cartan subalgebra,

$$[\mathbf{E}_3, \mathbf{E}_4] = f_{34}^1 \, \mathbf{H}_1 + f_{34}^2 \, \mathbf{H}_2 \; . \tag{7.65}$$

The facts that $[\mathbf{H}_1, \mathbf{E}_3] = \mathbf{0}$, and that $\mathrm{Ad}(\mathbf{E}_3)$ is antisymmetric with respect to the Cartan metric, imply that the commutator $[\mathbf{E}_3, \mathbf{E}_4]$ is orthogonal to $\mathbf{H}_1$. Proof:

$$\begin{aligned} (\mathbf{H}_1, [\mathbf{E}_3, \mathbf{E}_4]) &= (\mathbf{H}_1, \mathrm{Ad}(\mathbf{E}_3)\mathbf{E}_4) = -(\mathrm{Ad}(\mathbf{E}_3)\mathbf{H}_1, \mathbf{E}_4) \\ &= -([\mathbf{E}_3, \mathbf{H}_1], \mathbf{E}_4) = -(\mathbf{0}, \mathbf{E}_4) = 0 \; . \end{aligned} \tag{7.66}$$

Hence,

$$[\mathbf{E}_3, \mathbf{E}_4] = f_{34}^2 \, \mathbf{H}_2 \; . \tag{7.67}$$

Relations such as these are insufficient to determine all the structure constants uniquely, but the remaining ambiguities amount to just a choice of complex normalization factors for all the eigenvectors $\mathbf{E}_\alpha$ corresponding to different non-zero roots α. Thus, the root diagram identifies uniquely the complex simple Lie algebra $\mathrm{sl}(3, \mathbb{C})$.

7.5 SU(3) and quarks

A complex Lie group has a complex Lie algebra and is never compact, in the same way as the exponential e^z of a complex number $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$ is unbounded and can have an arbitrarily large absolute value $|\mathrm{e}^z| = \mathrm{e}^x$. The compact Lie group U(1) consists of the exponentials $\mathrm{e}^{-\mathrm{i}y}$ with $y \in \mathbb{R}$.

Thus, the compact group SU(3) is a real Lie group, and its Lie algebra su(3) is a real Lie algebra with the following eight anti-Hermitean traceless 3×3 matrices as a basis,

$$\begin{aligned} \boldsymbol{\lambda}_1 &= -\mathrm{i}\mathbf{E}_1 \; , \quad \boldsymbol{\lambda}_2 = -\mathrm{i}\mathbf{E}_2 \; , \quad \boldsymbol{\lambda}_3 = -\mathrm{i}(\mathbf{E}_3 + \mathbf{E}_4) \; , \quad \boldsymbol{\lambda}_4 = \mathbf{E}_3 - \mathbf{E}_4 \; , \\ \boldsymbol{\lambda}_5 &= -\mathrm{i}(\mathbf{E}_5 + \mathbf{E}_6) \; , \quad \boldsymbol{\lambda}_6 = \mathbf{E}_5 - \mathbf{E}_6 \; , \quad \boldsymbol{\lambda}_7 = -\mathrm{i}(\mathbf{E}_7 + \mathbf{E}_8) \; , \quad \boldsymbol{\lambda}_8 = \mathbf{E}_7 - \mathbf{E}_8 \; . \end{aligned} \tag{7.68}$$

A generator of SU(3) is a linear combination

$$\mathbf{L} = \sum_i a_i \boldsymbol{\lambda}_i \tag{7.69}$$

with real coefficients a_i. If we complexify su(3) by accepting complex coefficients a_i here, then we obtain the complex Lie algebra $\mathrm{sl}(3, \mathbb{C})$.

There are several different real Lie algebras having $\mathrm{sl}(3, \mathbb{C})$ as their complex extension. One is su(3), and another is $\mathrm{sl}(3, \mathbb{R})$. We let all of them be represented by the root diagram of $\mathrm{sl}(3, \mathbb{C})$.

In general, a complex simple Lie algebra has many real subalgebras, and among these there is exactly one that is the Lie algebra of a compact Lie group. Thus, the classification of the complex simple Lie algebras is at the same time a classification of the Lie algebras of compact simple Lie groups.

The quark model is an example of a physical application where SU(3) is an approximate symmetry group. A quantum mechanical observable is a Hermitean operator, and the observable corresponding to the SU(3) generator $\boldsymbol{\lambda}_j$ is $\mathrm{i}\boldsymbol{\lambda}_j$. In particular, we may choose to interpret the physical observables corresponding to the generators of the Cartan subalgebra as the third component of the isospin,

$$\mathbf{I}_3 = \frac{\mathrm{i}}{2}\,\boldsymbol{\lambda}_1 = \frac{1}{2}\,\mathbf{E}_1 = \sqrt{3}\,\mathbf{H}_1 \,, \tag{7.70}$$

and the hypercharge $Y = B + S$ where B is baryon number and S is strangeness,

$$\mathbf{Y} = \frac{\mathrm{i}}{3}\,\boldsymbol{\lambda}_2 = \frac{1}{3}\,\mathbf{E}_2 = 2\,\mathbf{H}_2 \,. \tag{7.71}$$

The electric charge is

$$\mathbf{Q} = \mathbf{I}_3 + \frac{1}{2}\,\mathbf{Y} \,. \tag{7.72}$$

These matrices are diagonal, and the common eigenvectors correspond to the quarks u, d, s,

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \qquad \mathbf{d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \qquad \mathbf{s} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{7.73}$$

If we denote by w_1 and w_2 the eigenvalues of the normalized Cartan matrices $\mathbf{H}_1 = \mathbf{I}_3/\sqrt{3}$ and $\mathbf{H}_2 = \mathbf{Y}/2$, then we may define a *weight vector* $w = (w_1, w_2)$ corresponding to the root vector $\alpha = (\alpha_1, \alpha_2)$. The eigenvalues are tabulated here,

	$\mathbf{u}$	$\mathbf{d}$	$\mathbf{s}$
w_1	$\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	0
w_2	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$

(7.74)

and are plotted in Figure 7.2. The difference between any two weight vectors is a root vector.

The antiquarks $\overline{u}, \overline{d}, \overline{s}$ belong to the complex conjugate representation of the Lie algebra su(3). Complex conjugation of matrices, $\mathbf{A} \to \mathbf{A}^*$, preserves both matrix multiplication,

$$\mathbf{W} = \mathbf{U}\mathbf{V} \quad \to \quad \mathbf{W}^* = \mathbf{U}^*\mathbf{V}^* \,, \tag{7.75}$$

and commutation,

$$\mathbf{N} = [\mathbf{L}, \mathbf{M}] \quad \to \quad \mathbf{N}^* = [\mathbf{L}^*, \mathbf{M}^*] \,. \tag{7.76}$$

The eight matrices $\boldsymbol{\lambda}_i$ define a Lie algebra with real structure constants. Hence, the complex conjugate matrices

$$\begin{aligned} \boldsymbol{\lambda}_1^* &= -\boldsymbol{\lambda}_1 \,, \quad \boldsymbol{\lambda}_2^* = -\boldsymbol{\lambda}_2 \,, \quad \boldsymbol{\lambda}_3^* = -\boldsymbol{\lambda}_3 \,, \quad \boldsymbol{\lambda}_4^* = \boldsymbol{\lambda}_4 \,, \\ \boldsymbol{\lambda}_5^* &= -\boldsymbol{\lambda}_5 \,, \quad \boldsymbol{\lambda}_6^* = \boldsymbol{\lambda}_6 \,, \quad \boldsymbol{\lambda}_7^* = -\boldsymbol{\lambda}_7 \,, \quad \boldsymbol{\lambda}_8^* = \boldsymbol{\lambda}_8 \end{aligned} \tag{7.77}$$

satisfy the same commutation relations as the matrices $\boldsymbol{\lambda}_i$, they define an inequivalent representation of the same Lie algebra. In this representation the isospin third component and the hypercharge are

$$\begin{aligned} \mathbf{I}_3 &= \frac{\mathrm{i}}{2}\,\boldsymbol{\lambda}_1^* = -\frac{1}{2}\,\mathbf{E}_1 = -\sqrt{3}\,\mathbf{H}_1 \,, \\ \mathbf{Y} &= \frac{\mathrm{i}}{3}\,\boldsymbol{\lambda}_2^* = -\frac{1}{3}\,\mathbf{E}_2 = -2\,\mathbf{H}_2 \,. \end{aligned} \tag{7.78}$$

The eigenvectors now have their physical interpretation as antiquarks,

$$\overline{\mathbf{u}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \overline{\mathbf{d}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \overline{\mathbf{s}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{7.79}$$

We consider the eigenvalues w_1, w_2 of redefined normalized Cartan matrices $\mathbf{H}_1' = \mathbf{I}_3/\sqrt{3} = -\mathbf{H}_1$ and $\mathbf{H}_2' = \mathbf{Y}/2 = -\mathbf{H}_2$, as tabulated here,

$$\begin{array}{c|ccc} & \overline{\mathbf{u}} & \overline{\mathbf{d}} & \overline{\mathbf{s}} \\ \hline w_1 & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 \\ w_2 & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{array} \tag{7.80}$$

The weight diagram of antiquarks is also plotted in Figure 7.2. Again, the difference between any two weight vectors is a root vector.

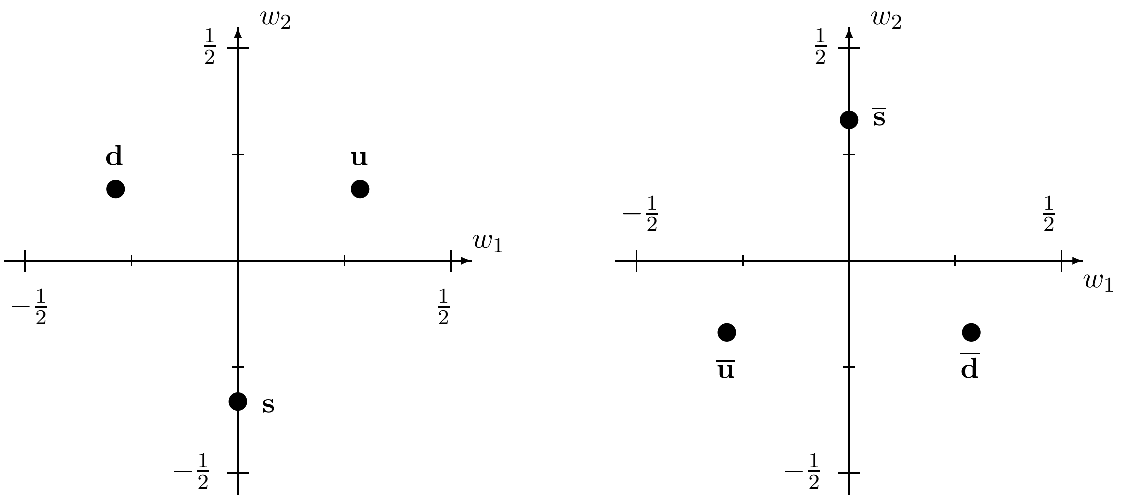

Figure 7.2: Weight diagram of quarks (left) and antiquarks (right).

These two three dimensional representations may be called 3 and $\overline{3}$, respectively. They are fundamental representations, since we may build up all irreducible representations of SO(3) by taking multiple tensor products of these two. A weight vector of a tensor product representation is a sum of weight vectors, one from each of the representations in the tensor product.

For example, a system of one quark and one antiquark has a nine dimensional representation which splits into two irreducible representations, one of dimension one and one of dimension eight. We write

$$3 \otimes \overline{3} = 1 \oplus 8 . \tag{7.81}$$

The singlet representation here has the weight vector $(0, 0)$. The octet representation is isomorphic to the adjoint representation, its weight diagram is identical to the root diagram in Figure 7.1.