FY 8104 Symmetry in physics

Problem 2, 2007

1. **Mini dictionary**

   Describe as briefly as you can the contents of the following concepts:

   (a) Transformation
   (b) Transformation group
   (c) Symmetry transformation
   (d) Cyclic group (Norwegian: syklisk gruppe)
   (e) The order of a group element (N: ordenen til et gruppeelement)
   (f) Commutative group (N: kommutativ)
   (g) Non-Abelian group (N: ikke-Abelsk)
   (h) Isomorphic groups (N: isomorfe grupper)
   (i) Homomorphism (N: homomorfi)
   (j) Left and right cosets (N: høyre og venstre restklasser)
   (k) Invariant subgroup (N: invariant undergruppe)
   (l) Factor group (N: faktorgruppe)
   (m) Direct product group (N: direkteproduktgruppe)
   (n) Conjugation class (N: konjugasjonsklasse)
   (o) Representation of a group (N: representasjon)
   (p) Equivalent representations of a group
   (q) Unitary representation of a group (N: unitær)
   (r) Irreducible representation of a group (N: irredusibel)

2. **Bits and pieces**

   (a) What is the inverse group element of gh (in terms of g^{-1} and h^{-1})?
   (b) The number of elements of a group (or subgroup) is called the *order* of the group. Every element h in a group of finite order defines a *cyclic* subgroup $H = \left\{h, h^2, \ldots, h^N = e\right\}$. The element h is said to be of order N if this subgroup is of order N. Show that all elements of a conjugation class are of the same order.
   (c) Show that a group G of prime number order must be equal to the cyclic (sub)group generated by any $g \neq e$ in G.

3. **Representations of $\mathbf{C}_{3v}$**

The transformation group $\mathbf{C}_{3v}$ (studied in Problem 1) may be defined to act on all points in the plane, so that the group element g transforms a point $\vec{x} \equiv (x, y)$ into $\vec{x}' = g(\vec{x}) \equiv (x', y')$, where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathcal{D}(g) \begin{pmatrix} x \\ y \end{pmatrix},$$

with $\mathcal{D}(g)$ a real 2×2-matrix.

(a) Choose the (x, y) coordinate system with origin at the centre of the triangle, and for example with the x-axis parallel to one edge. Find the matrix $\mathcal{D}(g)$ representing each element $g \in \mathbf{C}_{3v}$.
These matrices define one particular *linear representation* of the group $\mathbf{C}_{3v}$. It is a real representation, as opposed to a complex representation, since the coordinates (x, y) are real.

(b) The transformation group also has other linear representations, by acting on functions f defined on the (x, y) plane,

$$g : f \to f',$$

in such a way that $f'(\vec{x}') = f(\vec{x})$. In words: the transformed function f' has the same function value at the transformed point $\vec{x}'$ as the original function f has at the original point $\vec{x}$. This relation may also be written as $f'(\vec{x}') = f(g^{-1}(\vec{x}'))$. Since $\vec{x}'$ here is any arbitrary point in the plane, we may drop the prime on $\vec{x}'$ and write the last equation as $f'(\vec{x}) = f(g^{-1}(\vec{x}))$.
Show that this construction has the property of a representation of the group, that the result of operating with g_1 followed by g_2 leads to the same result as operating with $g_2 g_1$ directly. Remember that the order of successive transformations is read from right to left in the group product $g_2 g_1$.
This representation of the group as a transformation group acting on functions is a real representation if the function values are real numbers, or is a complex representation if the function values are complex numbers.

(c) Assume that $f(\vec{x})$ is a linear function of the general form $f(\vec{x}) = x\, a_1 + y\, a_2$.
After transformation by g we get a new linear function, $f'(\vec{x}) = x\, a_1' + y\, a_2'$.
Find for every $g \in \mathbf{C}_{3v}$ the relation between the coefficient vectors: $\vec{a}' = \mathcal{D}(g)\, \vec{a}$.
The 2×2-matrix $\mathcal{D}(g)$ here is not necessarily the same matrix as the one we introduced above.

(d) Assume that $f(\vec{x})$ is a quadratic function of the general form

$$f(\vec{x}) = (x^2 + y^2)\, b_0 + (x^2 - y^2)\, b_1 + (2xy)\, b_2 \; .$$

After transformation by g we get a new quadratic function,

$$f'(\vec{x}) = (x^2 + y^2)\, b_0' + (x^2 - y^2)\, b_1' + (2xy)\, b_2' \; .$$

Find for every $g \in \mathbf{C}_{3v}$ the relation between the coefficient vectors: $\vec{b}' = \mathcal{D}(g)\, \vec{b}$.
Here $\mathcal{D}(g)$ is a 3×3-matrix.