Chapter 3

Finite dimensional group representations

In the theory of linear representations of groups it is useful to distinguish between representations on finite and on infinite dimensional vector spaces. Another useful distinction is between finite and infinite groups, although the fundamental difference is rather between the finite groups and the compact infinite groups, on the one hand, and the non-compact infinite groups, on the other hand.

In this chapter and the next we will consider finite dimensional representations of groups, beginning with results that do not depend on whether the group is finite or infinite. In the next chapter we will restrict ourselves mostly to the case of representations of finite groups on finite dimensional complex vector spaces.

It is possible to study representations on vector spaces based on a general number field $\mathbb{F}$, and many results are valid in such a general setting. The cases of physical interest are mainly $\mathbb{F} = \mathbb{R}$, the real numbers, and $\mathbb{F} = \mathbb{C}$, the complex numbers. We will specialize here to complex representations, which are of special interest both from the mathematical and the physical point of view. It will usually be obvious which results depend on special properties of the complex numbers, and which results hold also for other number fields.

The great mathematical advantage of the complex numbers is the basic fact that they are algebraically complete, that every polynomial of degree n with complex coefficients has exactly n complex roots. This result, known as the *fundamental theorem of algebra*, simplifies parts of the theory. Complex representations are of physical interest because states of a physical system are represented in quantum mechanics as vectors in a complex Hilbert space. Real group representations also occur in physics, in order to understand them we will have to relate them to complex representations.

Definition 3.1 *A* "linear representation" *(or* "representation"*) ρ of the group G on the complex vector space V is a realization of G as a group of linear transformations on V. The group element g is represented as the linear transformation $\rho(g)$, and the group product is represented as composition of linear transformations, $\rho(gh) = \rho(g)\,\rho(h)$.*

The representation is "faithful" *if the function $\rho : g \mapsto \rho(g)$ is one to one.*

The function ρ should be continuous if G is a topological group, and differentiable (smooth) if G is a Lie group.

Two representations ρ on V and σ on W are "equivalent" *if there exists an invertible linear transformation $S : V \to W$ such that $\sigma(g) = S\rho(g)S^{-1}$ for all $g \in G$.*

A "matrix representation" $\mathbf{D}$ *represents every group element* g *as an* $n \times n$ *matrix* $\mathbf{D}(g)$*, in such a way that the group product becomes matrix multiplication,* $\mathbf{D}(gh) = \mathbf{D}(g)\,\mathbf{D}(h)$.

A matrix representation is just a special case of a linear representation, since a complex $n \times n$ matrix $\mathbf{D}(g)$ may be regarded as a linear transformation on $\mathbb{C}^n$.

Since a representation ρ is a group homomorphism from G into the group of invertible linear transformations on V, it follows from the general group theory that $\rho(e) = I$, the unit element $e \in G$ is represented by the identity transformation I on V. It also follows that $\rho(g^{-1}) = \rho(g)^{-1}$, where g and g^{-1} are inverse group elements, and $\rho(g)$ and $\rho(g)^{-1}$ are inverse linear transformations.

A representation ρ on V and an invertible linear transformation $S : V \to W$ always define an equivalent representation $S\rho S^{-1}$ on W. In particular, since an n dimensional complex vector space V is isomorphic to $\mathbb{C}^n$, a linear representation ρ on V is always equivalent to some representation by $n \times n$ matrices with complex matrix elements.

Equivalent representations are identical for classification purposes, and can all be identified with some standard matrix representation through the proper choice of basis. Thus, the problem of classifying all linear representations of a group G reduces to the problem of finding all the inequivalent matrix representations.

3.1 Matrix representations

To demonstrate explicitly how a general linear representation ρ is equivalent to a matrix representation $\mathbf{D}$, choose a basis $[\mathbf{e}] = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ in V. Every vector $\mathbf{v} \in V$ may be written in a unique way as a linear combination of the basis vectors,

$$\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{e}_i \ , \tag{3.1}$$

with complex coefficients a_i. The correspondence

$$\mathbf{v} \leftrightarrow \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \tag{3.2}$$

is an isomorphism between the vector spaces V and $\mathbb{C}^n$. Next, introduce $n \times n$ matrices

$$\mathbf{D}(g) = \begin{pmatrix} D_{11}(g) & D_{12}(g) & \ldots & D_{1n}(g) \\ D_{21}(g) & D_{22}(g) & \ldots & D_{2n}(g) \\ \vdots & \vdots & \ddots & \vdots \\ D_{n1}(g) & D_{n2}(g) & \ldots & D_{nn}(g) \end{pmatrix}, \tag{3.3}$$

with complex matrix elements $D_{ij}(g)$ defined by the relation

$$\rho(g)\,\mathbf{e}_i = \sum_{j=1}^{n} D_{ji}(g)\,\mathbf{e}_j \ . \tag{3.4}$$

Note that the summation index j here is the first index of the matrix.

The correspondence $\mathbf{v} \leftrightarrow \mathbf{a}$, eq. (3.2), implies that $\rho(g)\,\mathbf{v} \leftrightarrow \mathbf{D}(g)\,\mathbf{a}$, where $\mathbf{D}(g)\,\mathbf{a}$ is the matrix product of the $n \times n$ matrix $\mathbf{D}(g)$ and the $n \times 1$ matrix $\mathbf{a}$. The proof is simple,

$$\rho(g)\,\mathbf{v} = \sum_{i=1}^{n} a_i\,\rho(g)\,\mathbf{e}_i = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} D_{ji}(g)\,\mathbf{e}_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} D_{ji}(g)\,a_i\right)\mathbf{e}_j\ . \tag{3.5}$$

It follows that

$$\mathbf{D}(gh)\,\mathbf{a} \leftrightarrow \rho(gh)\,\mathbf{v} = (\rho(g)\,\rho(h))\,\mathbf{v} = \rho(g)\,(\rho(h)\,\mathbf{v}) \leftrightarrow \mathbf{D}(g)\,(\mathbf{D}(h)\,\mathbf{a}) = (\mathbf{D}(g)\,\mathbf{D}(h))\,\mathbf{a}\ . \tag{3.6}$$

The last equality holds because matrix multiplication is associative. Since the above relation holds for every $\mathbf{a} \in \mathbb{C}^n$, we conclude that

$$\mathbf{D}(gh) = \mathbf{D}(g)\,\mathbf{D}(h)\ . \tag{3.7}$$

We may give a second proof of this matrix equation using directly the definition (3.4),

$$\begin{aligned}\sum_j D_{ji}(gh)\,\mathbf{e}_j\ &=\ \rho(gh)\,\mathbf{e}_i = \rho(g)\,\rho(h)\,\mathbf{e}_i = \rho(g)\sum_k D_{ki}(h)\,\mathbf{e}_k = \sum_k D_{ki}(h)\,\rho(g)\,\mathbf{e}_k \\ &= \sum_k D_{ki}(h)\sum_j D_{jk}(g)\,\mathbf{e}_j = \sum_j\left(\sum_k D_{jk}(g)\,D_{ki}(h)\right)\mathbf{e}_j\ . \end{aligned}\tag{3.8}$$

We have omitted the summation limits here in order to save writing, and to make the formulae more easily readable. Eq. (3.8) proves eq. (3.7) as written explicitly in terms of matrix elements,

$$D_{ji}(gh) = \sum_k D_{jk}(g)\,D_{ki}(h)\ . \tag{3.9}$$

A linear transformation S on V has matrix elements S_{ij} defined by the expansion

$$S\mathbf{e}_i = \sum_j S_{ji}\,\mathbf{e}_j\ . \tag{3.10}$$

If S is invertible, then the n vectors $\widetilde{\mathbf{e}}_i = S\mathbf{e}_i$ are linearly independent and constitute a new basis in V. Relative to the new basis the representation ρ on V corresponds to a new matrix representation $\widetilde{\mathbf{D}}$ such that

$$\rho(g)\,\widetilde{\mathbf{e}}_i = \sum_j \widetilde{D}_{ji}(g)\,\widetilde{\mathbf{e}}_j\ . \tag{3.11}$$

The left hand side of this equation is

$$\begin{aligned}\rho(g)\,\widetilde{\mathbf{e}}_i\ &=\ \rho(g)\sum_j S_{ji}\,\mathbf{e}_j = \sum_j S_{ji}\,\rho(g)\,\mathbf{e}_j = \sum_j S_{ji}\sum_k D_{kj}(g)\,\mathbf{e}_k \\ &= \sum_k\sum_j D_{kj}(g)\,S_{ji}\,\mathbf{e}_k\ . \end{aligned}\tag{3.12}$$

The right hand side is

$$\sum_j \widetilde{D}_{ji}(g)\,\widetilde{\mathbf{e}}_j = \sum_j \widetilde{D}_{ji}(g)\sum_k S_{kj}\,\mathbf{e}_k = \sum_k\sum_j S_{kj}\,\widetilde{D}_{ji}(g)\,\mathbf{e}_k\ . \tag{3.13}$$

We see that the two matrix representations $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ corresponding to the same representation ρ are related by the equation

$$\sum_j D_{kj}(g)\, S_{ji} = \sum_j S_{kj}\, \widetilde{D}_{ji}(g) \ , \tag{3.14}$$

which we write in matrix notation as

$$\mathbf{D}(g)\,\mathbf{S} = \mathbf{S}\,\widetilde{\mathbf{D}}(g) \ . \tag{3.15}$$

Since the matrix $\mathbf{S}$ relating the two bases must be invertible, we have that

$$\widetilde{\mathbf{D}}(g) = \mathbf{S}^{-1}\,\mathbf{D}(g)\,\mathbf{S} \ . \tag{3.16}$$

A matrix transformation of the type $\mathbf{D}(g) \to \widetilde{\mathbf{D}}(g) = \mathbf{S}^{-1}\,\mathbf{D}(g)\,\mathbf{S}$, resulting from a change of basis, is called a *similarity transformation*.

3.2 Reducible and irreducible representations

Definition 3.2 *A subspace $U \subset V$ is* "invariant" *if $\rho(g)\mathbf{u} \in U$ for all $g \in G$ and $\mathbf{u} \in U$. An invariant subspace U is* "irreducible" *if it contains no invariant subspace W apart from the trivial ones, $W = U$ and $W = \{\mathbf{0}\}$.*

The representation ρ is "irreducible" *if the whole vector space V is irreducible, otherwise ρ is* "reducible"*.*

ρ is "fully reducible" *if it has two or more non-trivial complementary irreducible subspaces.*

By definition, subspaces $V_1, V_2, \ldots, V_k \subset V$ are *complementary* if every vector $\mathbf{v} \in V$ has a unique decomposition as

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k \qquad \text{with} \qquad \mathbf{v}_i \in V_i \ . \tag{3.17}$$

Note that the definition of invariant or irreducible subspaces has to be modified slightly if V is an infinite dimensional real or complex vector space. In that case the invariant subspaces we consider are further required to be *closed* with respect to some topology. If V is finite dimensional it has a standard topology, and every subspace is closed in this topology, so that the condition of closure need not be stated explicitly.

Note also that the reducibility or irreducibility of a group representation may depend on the number field $\mathbb{F}$. Thus, a representation by real matrices may well be irreducible when regarded as a real representation, but reducible when regarded as a complex representation. This is one of the reasons for restricting our discussion to the complex case.

There always exist the trivial invariant subspaces V and $\{\mathbf{0}\}$. More generally, each vector $\mathbf{v} \in V$ generates an invariant subspace, spanned by the vectors $\rho(g)\,\mathbf{v}$ which are the transforms of $\mathbf{v}$ by elements $g \in G$. This subspace is finite dimensional if G is finite. It need not be irreducible, if the representation ρ is reducible. It may be all of V, in which case $\mathbf{v}$ is called a *"generator"*, or *"cyclic vector"*, and ρ is a *"cyclic representation"*.

Theorem 3.3 *A group representation is irreducible if and only if every vector $\mathbf{v} \neq \mathbf{0}$ is cyclic. Hence, every irreducible representation of a finite group must be finite dimensional.*

Unitary or real orthogonal representations are always either irreducible or fully reducible, and the same is true for all representations of a group G which is either finite or compact. However, there are very simple counterexamples for infinite, non-compact groups. Take for example $G = \mathbb{Z}$, the additive group of integers, and let $a \in \mathbb{Z}$ be represented by the 2×2 matrix

$$\mathbf{D}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} . \tag{3.18}$$

It is easy to verify that $\mathbf{D}$ is a representation, that $\mathbf{D}(a)\,\mathbf{D}(b) = \mathbf{D}(a+b)$, and that there is one single non-trivial invariant subspace of $\mathbb{C}^2$,

$$U = \{ \begin{pmatrix} c \\ 0 \end{pmatrix} \mid c \in \mathbb{C} \} . \tag{3.19}$$

Since there is no invariant subspace complementary to U, the representation $\mathbf{D}$ is reducible but not fully reducible.

3.3 Direct sum and direct product

Direct sum and product of vector spaces

Let V and W be vector spaces of dimensions n and p, respectively.

Definition 3.4 *Given basis vectors* $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \in V$ *and* $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_p \in W$.
The "direct sum" $V \oplus W$ *is an* $n + p$ *dimensional vector space with basis vectors*

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n, \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_p .$$

The "direct product" *(or* "tensor product"*)* $V \otimes W$ *is an* np *dimensional vector space with basis vectors*

$$\mathbf{e}_1 \otimes \mathbf{f}_1, \ldots, \mathbf{e}_1 \otimes \mathbf{f}_p, \mathbf{e}_2 \otimes \mathbf{f}_1, \ldots, \mathbf{e}_2 \otimes \mathbf{f}_p, \ldots, \mathbf{e}_n \otimes \mathbf{f}_1, \ldots, \mathbf{e}_n \otimes \mathbf{f}_p .$$

Every vector $\mathbf{u} \in V \oplus W$ is the sum $\mathbf{u} = \mathbf{v} + \mathbf{w} = \mathbf{v} \oplus \mathbf{w}$ of two uniquely determined component vectors

$$\mathbf{v} = \sum_{i=1}^{n} a_i\, \mathbf{e}_i \in V , \qquad \mathbf{w} = \sum_{j=1}^{p} b_j\, \mathbf{f}_j \in W . \tag{3.20}$$

But not every vector in $V \otimes W$ is a direct product vector of the form

$$\mathbf{v} \otimes \mathbf{w} = \sum_{i=1}^{n} \sum_{j=1}^{p} a_i b_j\, \mathbf{e}_i \otimes \mathbf{f}_j . \tag{3.21}$$

In fact, the most general vector $\mathbf{u} \in V \otimes W$ can be expanded as

$$\mathbf{u} = \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ij}\, \mathbf{e}_i \otimes \mathbf{f}_j , \tag{3.22}$$

with np complex coefficients c_{ij}. The equation $\mathbf{u} = \mathbf{v} \otimes \mathbf{w}$ means that $c_{ij} = a_i b_j$ for all i, j, and solutions exist for a_i and b_j only if

$$c_{ij}\, c_{kl} = c_{il}\, c_{jk} \quad \text{for} \quad i, k = 1, \dots, n\ ;\ \ j, l = 1, \dots, p\ . \tag{3.23}$$

The above bases in V and W introduce the following isomorphisms,

$$\mathbf{v} \in V \leftrightarrow \mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{C}^n\ , \qquad \mathbf{w} \in W \leftrightarrow \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix} \in \mathbb{C}^p\ , \tag{3.24}$$

next,

$$\mathbf{v} \oplus \mathbf{w} \in V \oplus W \leftrightarrow \mathbf{a} \oplus \mathbf{b} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_p \end{pmatrix} \in \mathbb{C}^{n+p}\ , \tag{3.25}$$

and finally,

$$\mathbf{v} \otimes \mathbf{w} \in V \otimes W \leftrightarrow \mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_1\mathbf{b} \\ \vdots \\ a_n\mathbf{b} \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ \vdots \\ a_n b_p \end{pmatrix} \in \mathbb{C}^{np}\ . \tag{3.26}$$

Direct product Hilbert spaces are frequently encountered in quantum theory. A physical system is very often a *composite* system made up of two or more subsystems, for example two or more interacting particles. If there are two subsystems, and if the quantum states of the two subsystems are represented as vectors in the complex Hilbert spaces V and W, respectively, then the Hilbert space of the total system is $V \otimes W$.

In such a composite system any product state of the form $\mathbf{v} \otimes \mathbf{w}$ has the special property that there is no correlation between the results of measurements performed separately on the two subsystems. In all other states such correlations can be observed. If a state is not a product state, it is said to be *entangled*, because some measurements on the subsystems will show stronger correlations than what is possible according to classical physics.

Direct sum and product of operators

Given two linear operators A on V and B on W, the direct sum operator $A \oplus B$ and the direct product operator $A \otimes B$ act on $V \oplus W$ and on $V \otimes W$, respectively, so that

$$\begin{aligned} (A \oplus B)(\mathbf{v} \oplus \mathbf{w}) &= (A\mathbf{v}) \oplus (B\mathbf{w})\ , \\ (A \otimes B)(\mathbf{v} \otimes \mathbf{w}) &= (A\mathbf{v}) \otimes (B\mathbf{w})\ . \end{aligned} \tag{3.27}$$

Relative to the bases defined above, the operator A is represented by an $n \times n$ matrix $\mathbf{A}$, B by a $p \times p$ matrix $\mathbf{B}$, $A \oplus B$ by the $(n+p) \times (n+p)$ matrix

$$\mathbf{A} \oplus \mathbf{B} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = \begin{pmatrix} A_{11} & \dots & A_{1n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} & 0 & \dots & 0 \\ 0 & \dots & 0 & B_{11} & \dots & B_{1p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & B_{p1} & \dots & B_{pp} \end{pmatrix} , \tag{3.28}$$

and $A \otimes B$ by the $(np) \times (np)$ matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} A_{11}\mathbf{B} & \dots & A_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ A_{n1}\mathbf{B} & \dots & A_{nn}\mathbf{B} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & \dots & A_{1n}B_{1p} \\ A_{11}B_{11} & A_{11}B_{12} & \dots & A_{1n}B_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B_{p1} & A_{n1}B_{p2} & \dots & A_{nn}B_{pp} \end{pmatrix} . \tag{3.29}$$

Direct sum and product of representations

Definition 3.5 *Given two linear representations of the group G, ρ on the vector space V and σ on W. The* "direct sum" $\tau = \rho \oplus \sigma$ *and* "direct product" $\pi = \rho \otimes \sigma$ *are representations on $V \oplus W$ and $V \otimes W$, respectively, such that*

$$\tau(g) = \rho(g) \oplus \sigma(g), \qquad \pi(g) = \rho(g) \otimes \sigma(g) \qquad \forall g \in G . \tag{3.30}$$

A fully reducible representation is the direct sum of its irreducible components, and it is therefore completely described by an enumeration of the irreducible representations it contains. Obviously, it may contain the same irreducible representation several times. In statements of this kind equivalent representations are always counted as identical.

More precisely, given a representation ρ of the group G on the vector space V, and given a set of complementary invariant subspaces $V_1, V_2, \dots, V_k \subset V$. Each subspace V_i is assumed to be invariant, but not necessarily irreducible. Then the restriction of ρ to one subspace V_i is a representation, reducible or irreducible, which we will call ρ_i. That is, we define

$$\rho_i(g)\,\mathbf{v} = \rho(g)\,\mathbf{v} \qquad \forall g \in G, \mathbf{v} \in V_i , \tag{3.31}$$

but $\rho_i(g)\,\mathbf{v}$ is undefined for $\mathbf{v} \notin V_i$. With this definition ρ is the direct sum,

$$\rho = \rho_1 \oplus \rho_2 \oplus \dots \oplus \rho_k . \tag{3.32}$$

Let n_i be the dimension of V_i, and choose a basis for V such that the first n_1 basis vectors are in V_1, the next n_2 in V_2, and so on. Relative to this basis any element $g \in G$ will have a matrix representation $\mathbf{D}(g)$ which is "block diagonal", that is,

$$\mathbf{D}(g) = \begin{pmatrix} \mathbf{D}_1(g) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2(g) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{D}_k(g) \end{pmatrix} , \tag{3.33}$$

with zeros everywhere outside the matrix blocks of dimension $n_i \times n_i$ on the diagonal. If every representation ρ_i is irreducible, then this is as near as it is possible to come to a simultaneous diagonalization of all the representation matrices $\mathbf{D}(g)$. The basis may always be chosen so that two submatrices $\mathbf{D}_i(g)$ and $\mathbf{D}_j(g)$ are identical whenever the two representations are equivalent.

Recall the definition, that $V_1, V_2, \ldots, V_k$ are complementary subspaces of V when every vector $\mathbf{v} \in V$ has a unique decomposition as

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k \qquad \text{with} \qquad \mathbf{v}_i \in V_i \; . \tag{3.34}$$

Every subspace V_i must contain the zero vector $\mathbf{0}$, and the unique decomposition of $\mathbf{0}$ is $\mathbf{0} = \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0}$. Assume that $\mathbf{u} \in V_i \cap V_j$ with $i \neq j$. Because $\mathbf{0} = \mathbf{u} + (-\mathbf{u})$ with $\mathbf{u} \in V_i$ and $-\mathbf{u} \in V_j$, and because the decomposition of $\mathbf{0}$ is unique, we must have $\mathbf{u} = \mathbf{0}$. This shows that $V_i \cap V_j = \{\mathbf{0}\}$.

By definition, a projection operator, or projection, P, is idempotent, $P^2 = P$. Defining $P_i\mathbf{v} = \mathbf{v}_i$, we get linear projection operators $P_1, P_2, \ldots, P_k$ such that $P_iV = V_i$. The projections onto the complementary subspaces $V_1, V_2, \ldots, V_k$ form a *decomposition (or resolution) of the identity.* That is, ${P_i}^2 = P_i$ for $i = 1, 2, \ldots, k$, $P_iP_j = 0$ when $i \neq j$, and

$$I = P_1 + P_2 + \cdots + P_k \; . \tag{3.35}$$

To summarize, complementary subspaces $V_1, V_2, \ldots, V_k$ define projections $P_1, P_2, \ldots, P_k$ that form a decomposition of the identity. And conversely, a decomposition of the identity define complementary subspaces $V_i = P_iV$.

Theorem 3.6 *A representation ρ of the group G is a direct sum as in eq. (3.32) if and only if there exists a decomposition of the identity as in eq. (3.35), such that every projection P_i commutes with ρ. That is, $P_i\,\rho(g) = \rho(g)\,P_i \; \forall g \in G$, $i = 1, 2, \ldots, k$.*

Proof. First the "if" part. Given a decomposition of the identity such that every P_i commutes with ρ. For $g \in G$ define the following operator on each of the complementary subspaces $V_i = P_iV$,

$$\rho_i(g) = P_i\,\rho(g)\,P_i = {P_i}^2\,\rho(g) = P_i\,\rho(g) = \rho(g)\,P_i \; . \tag{3.36}$$

We see that ρ is a direct sum, because

$$\rho(g) = \rho(g)\,I = \rho(g)\sum_{i=1}^{k} P_i = \sum_{i=1}^{k} \rho_i(g) \; . \tag{3.37}$$

And then the "only if" part. Given complementary invariant subspaces $V_1, \ldots, V_k \subset V$, not necessarily irreducible, and the corresponding decomposition of the unity, $I = P_1 + \cdots + P_k$, with $V_i = P_iV$. If $i \neq j$, we must have for $g \in G$ that

$$P_i\,\rho(g)\,P_j = 0 \; . \tag{3.38}$$

In fact, for every $\mathbf{v} \in V$ we have $P_j\,\mathbf{v} \in V_j$, $\rho(g)\,P_j\,\mathbf{v} \in V_j$, and $P_i\,\rho(g)\,P_j\,\mathbf{v} = 0$, since $P_i\,\mathbf{v}_j = \mathbf{0}$ whenever $\mathbf{v}_j \in V_j$ with $i \neq j$. It follows that every P_i commutes with ρ, in fact,

$$\begin{aligned} P_i\,\rho(g) \;&=\; P_i\,\rho(g)\,I = P_i\,\rho(g)\sum_{j=1}^{k} P_j = P_i\,\rho(g)\,P_i \\ &= \sum_{j=1}^{k} P_j\,\rho(g)\,P_i = I\,\rho(g)\,P_i = \rho(g)\,P_i\;. \end{aligned} \tag{3.39}$$

QED

Note that as soon as we have one nontrivial projection P ($P \neq 0$ and $P \neq I$) that commutes with a representation ρ, then we know that ρ is a direct sum, because the definition $Q = I - P$ gives us a second projection commuting with ρ. We verify directly that

$$Q^2 = (I-P)^2 = I - 2P + P^2 = I - P = Q\;, \tag{3.40}$$

and furthermore that $P + Q = I$ and $PQ = QP = P - P^2 = 0$.

3.4 Schur's lemma

Schur's lemma is very useful both in the general theory of group representations and in physical applications. A quantum mechanical problem where there is a non-trivial symmetry group can often be very much simplified by means of Schur's lemma, or by means of the Wigner–Eckart theorem, which is closely related.

A *division algebra* is an algebra with multiplicative unit element I in which every nonzero element A has a multiplicative inverse A^{-1} such that $A^{-1}A = AA^{-1} = I$. In particular, the only complex division algebra is the one consisting of the multiples of the identity, cI with $c \in \mathbb{C}$. See Appendix A.

Theorem 3.7 (Schur's lemma) *Let ρ and σ be irreducible linear representations of the group G over the (finite dimensional) vector spaces V and W.*

If $A : V \to W$ is a linear transformation such that $A\rho(g) = \sigma(g)A\ \forall g \in G$, then either $A = 0$, or A is invertible so that ρ and σ are equivalent, $\sigma(g) = A\rho(g)A^{-1}$.

The commutant $\rho(G)'$ of ρ, defined as the set of all linear transformations on V commuting with ρ,

$$\rho(G)' = \{\, B : V \to V \mid B\rho(g) = \rho(g)B \quad \forall g \in G \,\}\;, \tag{3.41}$$

is a division algebra. In particular, when ρ is an irreducible complex representation, then

$$\rho(G)' = \{\, cI \mid c \in \mathbb{C} \,\}\;. \tag{3.42}$$

In other words, the only linear transformations commuting with $\rho(g)$ for every $g \in G$ are the multiples of the identity I.

Proof. In the first part of the theorem, the image of V under A,

$$\operatorname{Img} A = \{\, A\mathbf{v} \mid \mathbf{v} \in V \,\}\;, \tag{3.43}$$

is an invariant subspace of W, since $\sigma(g)A\mathbf{v} = A\rho(g)\mathbf{v} \in \operatorname{Img} A$ for all $g \in G$. Since σ is irreducible, either $\operatorname{Img} A = \{\mathbf{0}\}$ or $\operatorname{Img} A = W$. Similarly, the kernel of A,

$$\operatorname{Ker} A = \{ \mathbf{v} \in V \mid A\mathbf{v} = \mathbf{0} \} , \tag{3.44}$$

is an invariant subspace of V. For if $A\mathbf{v} = \mathbf{0}$ and $g \in G$, then $A\rho(g)\mathbf{v} = \sigma(g)A\mathbf{v} = \mathbf{0}$. Since ρ is irreducible, either $\operatorname{Ker} A = \{\mathbf{0}\}$ or $\operatorname{Ker} A = V$.

Therefore, either $\operatorname{Ker} A = V$ and $\operatorname{Img} A = \{\mathbf{0}\}$ so that $A = 0$, or $\operatorname{Ker} A = \{\mathbf{0}\}$ and $\operatorname{Img} A = W$ so that A is invertible.

The last part of the theorem follows when we take $\sigma = \rho$. For example, it is obvious that $\rho(G)'$ is an algebra containing the identity I.

QED

An operator $A : V \to W$ such that $A\rho = \sigma A$ is called an *intertwining operator* of the representations ρ and σ. We say that it commutes with the action of the group G on V and on W.

It follows from Schur's lemma that if $A, B : V \to W$ are both invertible and commute with the action of G, so that $A\rho(g) = \sigma(g)A$ and $B\rho(g) = \sigma(g)B$ for all $g \in G$, then $A^{-1}B \in \rho(G)'$, hence in the complex case $A^{-1}B = cI$ and $B = cA$ for some complex number c.

Another immediate consequence is the following.

Theorem 3.8 *Every finite dimensional irreducible complex representation ρ of an Abelian group G is one dimensional.*

In fact, in this case every $\rho(g)$ belongs to the commutant $\rho(G)'$, and so $\rho(g) = c(g)\, I$ with $c(g) \in \mathbb{C}$. Then every one dimensional subspace of V is invariant.

Note that the group structure plays no part at all in the proof of Schur's lemma, the role of the group G is only that the group element g defines a correspondence between the linear transformations $\rho(g)$ on V and $\sigma(g)$ on W. Thus we have actually proved the more general result that any set X of linear transformations on V which leaves no proper subspace invariant, has a division algebra as its commutant, and in the complex case only multiples of the identity commute with every transformation in X.

Theorem 3.9 *The commutant of a direct sum of two (equivalent or inequivalent) group representations is an algebra of dimension at least two, and it is not a division algebra.*

Proof. The direct sum $(aI) \oplus (bI)$ where $a, b \in \mathbb{C}$ commutes with every $\rho(g) \oplus \sigma(g)$ where ρ and σ are representations of the group G, and $g \in G$. In particular, $0 \oplus I$ and $I \oplus 0$ are non-invertible non-zero elements of the commutant.

QED

This result is as close as we can get to a converse of Schur's lemma, that a group representation is irreducible if its commutant is a division algebra. As we shall see, all representations of finite groups and compact topological groups, as well as all unitary representations of any group, are either irreducible or fully reducible, and then the converse of Schur's lemma holds.

Theorem 3.10 *Assume that ρ is a complex representation of the group G, and that G is finite or compact, or G is non-compact but ρ is unitary.*

Then ρ is irreducible if the commutant $\rho(G)'$ consists of multiples of the identity.

A simple counterexample is the infinite and non-compact group of triangular 2×2 matrices of the form

$$\mathbf{D}(a,b,c) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} , \tag{3.45}$$

with complex parameters a, b, c such that

$$\det \mathbf{D}(a,b,c) = ac \neq 0 \; . \tag{3.46}$$

This matrix representation is reducible but not fully reducible, and its commutant consists of the multiples of the identity matrix.

3.5 Unitary and orthogonal representations

Unitary and orthogonal operators

A Hermitean scalar product on the complex vector space V assigns to every pair of vectors $\mathbf{u}, \mathbf{v} \in V$ a complex number $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})^*$. It is a linear function of its second argument,

$$(\mathbf{u}, a\mathbf{v} + b\mathbf{w}) = a\,(\mathbf{u}, \mathbf{v}) + b\,(\mathbf{u}, \mathbf{v}) \qquad \forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V,\ a, b \in \mathbb{C} \; . \tag{3.47}$$

Hence, it is antilinear in its first argument,

$$(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a^*\,(\mathbf{u}, \mathbf{w}) + b^*\,(\mathbf{v}, \mathbf{w}) \; . \tag{3.48}$$

We will assume here that the scalar product is positive definite, that $(\mathbf{u}, \mathbf{u}) > 0$ for all $\mathbf{u} \neq \mathbf{0}$. A complex vector space with a Hermitean positive definite scalar product is called a complex Hilbert space.

If A is an operator (a linear transformation) on the complex Hilbert space V, the adjoint, or Hermitean conjugate, operator $A^\dagger$ is defined by the condition that

$$(\mathbf{u}, A^\dagger \mathbf{v}) = (A\mathbf{u}, \mathbf{v}) \qquad \forall\, \mathbf{u}, \mathbf{v} \in V \; . \tag{3.49}$$

We say that A is self-adjoint, or Hermitean, if $A^\dagger = A$ (in the infinite dimensional case there is a subtle distinction between self-adjoint and Hermitean operators). We say that A is unitary if it is invertible and $A^\dagger = A^{-1}$. A unitary operator A preserves the scalar product,

$$(A\mathbf{u}, A\mathbf{v}) = (A^\dagger A\mathbf{u}, \mathbf{v}) = (A^{-1}A\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{v}) \; . \tag{3.50}$$

A real Hilbert space is a real vector space with a positive definite scalar product. In this case the scalar product is real valued and symmetric, $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$. The condition

$$(\mathbf{u}, A^\top \mathbf{v}) = (A\mathbf{u}, \mathbf{v}) \qquad \forall\, \mathbf{u}, \mathbf{v} \in V \tag{3.51}$$

defines the transpose $A^\top$ of an operator A on the real Hilbert space V. We say that A is symmetric if $A^\top = A$, and that A is orthogonal if it is invertible and $A^\top = A^{-1}$. Like a unitary operator on a complex Hilbert space, an orthogonal operator on a real Hilbert space preserves the scalar product.

Note that the term "orthogonal" is sometimes used also for an operator on a complex vector space if it preserves a scalar product which is not positive definite, but is symmetric, such that $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$, and is complex linear in both factors. To avoid confusion one may speak of "real orthogonal" and "complex orthogonal" operators.

Orthonormal bases

A basis $[\mathbf{e}] = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ for the n dimensional complex Hilbert space V is orthonormal if

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} \qquad \forall\, i, j = 1, 2, \ldots, n \ . \tag{3.52}$$

The matrix elements of an operator A relative to the basis $[\mathbf{e}]$ is defined in general by the relation

$$A\mathbf{e}_i = \sum_{j=1}^{n} A_{ji} \mathbf{e}_j \ . \tag{3.53}$$

An equivalent definition for an orthonormal basis is that

$$A_{ji} = (\mathbf{e}_j, A\mathbf{e}_i) \ . \tag{3.54}$$

Then the Hermitean conjugate operator $A^\dagger$ has the matrix elements

$$(A^\dagger)_{ji} = (\mathbf{e}_j, A^\dagger \mathbf{e}_i) = (A\mathbf{e}_j, \mathbf{e}_i) = (\mathbf{e}_i, A\mathbf{e}_j)^* = (A_{ij})^* \ . \tag{3.55}$$

Thus, when the operator A on V is represented, relative to an orthonormal basis, by the $n \times n$ matrix $\mathbf{A}$, then the Hermitean conjugate operator $A^\dagger$ is represented by the Hermitean conjugate matrix $\mathbf{A}^\dagger$, obtained from $\mathbf{A}$ by transposition and complex conjugation. In particular, a Hermitean operator A, with $A^\dagger = A$, is represented by a Hermitean matrix $\mathbf{A}$, with $\mathbf{A}^\dagger = \mathbf{A}$, and a unitary operator A, with $A^\dagger = A^{-1}$, is represented by a unitary matrix $\mathbf{A}$, with $\mathbf{A}^\dagger = \mathbf{A}^{-1}$.

Similarly, when we define matrix elements relative to an orthonormal basis in a real Hilbert space, the transposition of operators, $A \to A^\top$, corresponds to the transposition of matrices, $\mathbf{A} \to \mathbf{A}^\top$. In this case, a symmetric operator A, with $A^\top = A$, is represented by a symmetric matrix $\mathbf{A}$, with $\mathbf{A}^\top = \mathbf{A}$, and an orthogonal operator A, with $A^\top = A^{-1}$, is represented by an orthogonal matrix $\mathbf{A}$, with $\mathbf{A}^\top = \mathbf{A}^{-1}$.

The Gram–Schmidt orthogonalization procedure

It is always possible to choose an orthonormal basis in a complex or real Hilbert space, for example, starting from an arbitrary basis $[\mathbf{f}]$ and constructing an orthonormal basis $[\mathbf{e}]$ by the Gram–Schmidt procedure.

Let $\mathbf{e}_1 = a_1 \mathbf{f}_1$, where a_1 is a (complex or real) normalization factor chosen in such a way that

$$(\mathbf{e}_1, \mathbf{e}_1) = |a_1|^2 \, (\mathbf{f}_1, \mathbf{f}_1) = 1 \ . \tag{3.56}$$

Then for $k = 2, 3, \ldots, n$ define

$$\mathbf{e}_k = a_k \left(\mathbf{f}_k - \sum_{i=1}^{k-1} (\mathbf{e}_i, \mathbf{f}_k) \, \mathbf{e}_i \right) , \tag{3.57}$$

where a_k is a normalization factor chosen such that $(\mathbf{e}_k, \mathbf{e}_k) = 1$.

Unitary and orthogonal representations

Definition 3.11 *A complex linear representation* ρ *of the group* G *is* "unitary" *if the representation space* V *is a complex Hilbert space (either finite or infinite dimensional), and if* $\rho(g)$ *is unitary for every* $g \in G$. *Then*

$$(\rho(g)\mathbf{u}, \rho(g)\mathbf{v}) = (\mathbf{u}, \mathbf{v}) \qquad \forall\, \mathbf{u}, \mathbf{v} \in V,\ g \in G\ . \tag{3.58}$$

A real linear representation is "orthogonal" *if the representation space is a real Hilbert space, and if* $\rho(g)$ *is orthogonal for every* $g \in G$.

Theorem 3.12 *Let* ρ *be a unitary representation of* G *on* V, *and let* U *be an invariant subspace. The orthogonal complement*

$$U^\perp = \{\mathbf{v} \mid (\mathbf{u}, \mathbf{v}) = 0 \quad \forall \mathbf{u} \in U\} \tag{3.59}$$

is then a complementary invariant subspace.

Proof. Let $g \in G$ and $\mathbf{v} \in U^\perp$. We want to prove that $\rho(g)\mathbf{v} \in U^\perp$. For every $\mathbf{u} \in U$ we have that $\rho(g^{-1})\mathbf{u} \in U$, and

$$(\mathbf{u}, \rho(g)\mathbf{v}) = ((\rho(g))^\dagger \mathbf{u}, \mathbf{v}) = ((\rho(g))^{-1}\mathbf{u}, \mathbf{v}) = (\rho(g^{-1})\mathbf{u}, \mathbf{v}) = 0\ . \tag{3.60}$$

QED

Theorem 3.13 *Every (finite dimensional) reducible unitary representation is fully reducible.*

Proof. Let ρ be a unitary representation of G on V. If it is reducible, there exists a non-trivial invariant subspace U, and U and $U^\perp$ are complementary invariant subspaces. If U and/or $U^\perp$ are reducible, they can be split further in a similar way. In the finite dimensional case irreducible components will be obtained in a finite number of steps.

QED

These two theorems hold also for real orthogonal representations, by the same proofs.

3.6 Characters

Definition 3.14 *The* "character" *of a linear representation* ρ *of the group* G *is the function* $\chi : G \to \mathbb{C}$ *such that*

$$\chi(g) = \mathrm{Tr}\,\rho(g) \qquad \forall g \in G\ . \tag{3.61}$$

The character of the unit element e is just the dimension n of the representation, $\chi(e) = \mathrm{Tr}\,\rho(e) = \mathrm{Tr}\,I = n$.

In order to compute the trace we need a basis $[\mathbf{e}] = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ for the vector space V, turning $\rho(g)$ into an $n \times n$ matrix $\mathbf{D}(g)$, by eq. (3.4). Then

$$\mathrm{Tr}\,\rho(g) = \mathrm{Tr}\,\mathbf{D}(g) = \sum_{i=1}^{n} D_{ii}(g)\ . \tag{3.62}$$

The trace is actually basis independent, even though we use a basis to compute it. In fact, we have seen that a change of basis corresponds to a similarity transformation, $\mathbf{D}(g) \to \mathbf{S}^{-1}\,\mathbf{D}(g)\,\mathbf{S}$, and this leaves the trace invariant. Using the property of the trace that

$$\mathrm{Tr}(\mathbf{AB}) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} = \mathrm{Tr}(\mathbf{BA}) , \tag{3.63}$$

we find that

$$\mathrm{Tr}(\mathbf{S}^{-1}\mathbf{AS}) = \mathrm{Tr}((\mathbf{S}^{-1}\mathbf{A})\mathbf{S}) = \mathrm{Tr}(\mathbf{S}(\mathbf{S}^{-1}\mathbf{A})) = \mathrm{Tr}((\mathbf{SS}^{-1})\mathbf{A}) = \mathrm{Tr}(\mathbf{A}) . \tag{3.64}$$

Given an invertible linear transformation $S : V \to W$ we have a representation $\sigma = S\rho S^{-1}$ on W, equivalent to ρ on V. We have then that

$$\mathrm{Tr}\,\sigma(g) = \mathrm{Tr}(S\rho(g)S^{-1}) = \mathrm{Tr}(S^{-1}S\rho(g)) = \mathrm{Tr}\,\rho(g) . \tag{3.65}$$

Theorem 3.15 *Equivalent representations have identical characters.*

The converse statement, that equality of characters implies equivalence, again is not true in general, although it is true for irreducible or fully reducible representations, such as complex representations of finite and compact groups. As a counterexample we need an infinite group, let us consider the additive group of the integers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$. It has for example the two matrix representations

$$\mathbf{D}_1(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} , \qquad \mathbf{D}_2(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \text{where} \qquad a \in \mathbb{Z} . \tag{3.66}$$

We may verify that $\mathbf{D}_1(a)\mathbf{D}_1(b) = \mathbf{D}_1(a+b)$ and $\mathbf{D}_2(a)\mathbf{D}_2(b) = \mathbf{D}_2(a+b)$. These two representations have the same character, $\chi_1(a) = \chi_2(a) = 2$, but they are not equivalent.

Theorem 3.16 *The character is a* class function*. That is, it is a function only of the conjugation class to which a group element belongs.*

Proof. If g and h are conjugate group elements, $h = fgf^{-1}$, then

$$\chi(h) = \mathrm{Tr}\,\rho(h) = \mathrm{Tr}(\rho(f)\rho(g)\rho(f^{-1})) = \mathrm{Tr}(\rho(f^{-1})\rho(f)\rho(g)) = \mathrm{Tr}\,\rho(g) = \chi(g) . \tag{3.67}$$

QED

We have seen two ways of building new representations from old. Given two representations ρ_1 and ρ_2, we may construct the direct sum $\rho_s = \rho_1 \oplus \rho_2$ and the direct product $\rho_p = \rho_1 \otimes \rho_2$. The characters of these new representations are the sum and product of the characters of ρ_1 and ρ_2,

$$\chi_s(g) = \chi_1(g) + \chi_2(g) \qquad \chi_p(g) = \chi_1(g)\,\chi_2(g) . \tag{3.68}$$

3.7 The contragredient representation

From a matrix representation $\mathbf{D}$ we may construct the *contragredient*, or *dual*, representation $\overline{\mathbf{D}}$, defined by the relation $\overline{\mathbf{D}}(g) = (\mathbf{D}(g^{-1}))^\top$, or explicitly in terms of matrix elements,

$$\overline{D}_{ij}(g) = D_{ji}(g^{-1}) \ . \tag{3.69}$$

Because of the general relation $(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$, we see that $\overline{\mathbf{D}}$ is a representation,

$$\overline{\mathbf{D}}(gh) = (\mathbf{D}((gh)^{-1}))^\top = (\mathbf{D}(h^{-1})\mathbf{D}(g^{-1}))^\top = (\mathbf{D}(g^{-1}))^\top (\mathbf{D}(h^{-1}))^\top = \overline{\mathbf{D}}(g)\overline{\mathbf{D}}(h) \ . \tag{3.70}$$

When $\mathbf{D}$ is a unitary matrix representation, so that $\mathbf{D}(g^{-1}) = (\mathbf{D}(g))^{-1} = (\mathbf{D}(g))^\dagger$, then $\overline{\mathbf{D}}$ is just the complex conjugate of $\mathbf{D}$, in fact, $\overline{\mathbf{D}}(g) = (\mathbf{D}(g^{-1}))^\top = ((\mathbf{D}(g))^\dagger)^\top = (\mathbf{D}(g))^*$.

The two representations $\mathbf{D}$ and $\overline{\mathbf{D}}$ are often equivalent. A necessary condition for equivalence is that they have the same character, that is,

$$\mathrm{Tr}\,\mathbf{D}(g) = \mathrm{Tr}\,\overline{\mathbf{D}}(g) = \mathrm{Tr}\,\mathbf{D}(g^{-1}) \qquad \forall g \in G \ . \tag{3.71}$$

If $\mathbf{D}$ is irreducible or fully reducible, as is always the case for example if G is a finite group, then this condition is also sufficient. If for example $G = S_n$, the symmetric group, where g^{-1} is always conjugate to g, $\mathbf{D}$ and $\overline{\mathbf{D}}$ always have the same character and are equivalent.

Chapter 4

Complex representations of finite groups

In the linear representation theory for finite groups only representations on finite dimensional vector spaces are of interest. Every cyclic and therefore every irreducible representation of a group G of finite order $|G| = N$ is finite dimensional, of dimension at most equal to N. As we shall see immediately below, every representation of a finite group, at least if it is finite dimensional, is a direct sum of irreducible representations. Hence, in order to classify all possible finite dimensional representations, it is sufficient to classify all the irreducible representations. This is our purpose in the present chapter.

Most of the time we will consider matrix representations, since we do not distinguish between equivalent representations, and since every representation is equivalent to a matrix representation. When $\mathbf{D}$ is an n dimensional matrix representation, the group element g is represented by the $n \times n$ matrix $\mathbf{D}(g)$, having n^2 matrix elements $D_{ij}(g) \in \mathbb{C}$.

We will choose more or less arbitrarily a complete set of standard irreducible matrix representations, which we will denote by $\mathbf{D}^{(\mu)}$, with $\mu = 1, 2, \ldots, M$. Completeness means that every irreducible representation is equivalent to some $\mathbf{D}^{(\mu)}$, and it is to be understood that $\mathbf{D}^{(\mu)}$ and $\mathbf{D}^{(\nu)}$ are inequivalent when $\mu \neq \nu$. The dimension of $\mathbf{D}^{(\mu)}$ we denote by n_μ. As we shall see, the number M of inequivalent irreducible representations is finite and equal to K, the number of conjugation classes in G.

4.1 Unitarity and reducibility

Theorem 4.1 *Every complex representation of a finite group is unitary with respect to some positive definite scalar product. Hence it is either irreducible or fully reducible.*

Proof. Given the group G and a representation ρ over V. There always exists a positive definite scalar product $((\cdot,\cdot))$ on V, invariant or not. For example, choose an arbitrary basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ and define $((\mathbf{e}_i, \mathbf{e}_j)) = \delta_{ij}$. Then define

$$(\mathbf{u}, \mathbf{v}) = \sum_{g \in G} ((\rho(g)\mathbf{u}, \rho(g)\mathbf{v})) \qquad \forall\, \mathbf{u}, \mathbf{v} \in V\ . \tag{4.1}$$

This is now our invariant positive definite scalar product. It is invariant under $h \in G$ because

h just permutes the terms in the defining sum,

$$\begin{aligned}(\rho(h)\mathbf{u},\rho(h)\mathbf{v}) &= \sum_{g\in G}((\rho(g)\rho(h)\mathbf{u},\rho(g)\rho(h)\mathbf{v})) \\ &= \sum_{g}((\rho(gh)\mathbf{u},\rho(gh)\mathbf{v})) = \sum_{f}((\rho(f)\mathbf{u},\rho(f)\mathbf{v})) = (\mathbf{u},\mathbf{v})\ ,\end{aligned} \tag{4.2}$$

where $f = gh$. To prove that it is a positive definite scalar product, we have to prove for example that $(\mathbf{v},\mathbf{v}) > 0$ when $\mathbf{v} \neq \mathbf{0}$. We have that

$$(\mathbf{v},\mathbf{v}) = \sum_{g\in G}((\rho(g)\mathbf{v},\rho(g)\mathbf{v})) \geq ((\rho(e)\mathbf{v},\rho(e)\mathbf{v})) = ((\mathbf{v},\mathbf{v})) > 0\ . \tag{4.3}$$

Other required properties are equally easily demonstrated.

QED

This theorem implies that we know all complex representations of a finite group as soon as we know the finite dimensional irreducible representations. The theorem holds also for continuous representations of infinite compact topological groups, by a similar proof. The difference is that the sum over group elements must be replaced by an integral over the group, called the invariant Haar integral.

Since we may always choose an orthonormal basis in the Hilbert space, we have now actually proved the following result.

Theorem 4.2 *Every complex representation of a finite or compact group is equivalent to a unitary matrix representation.*

In the same way, every real representation of a finite or compact group is equivalent to a real orthogonal matrix representation.

We will make use of this theorem to assume that our standard irreducible representations $\mathbf{D}^{(\mu)}$ are unitary, that $(\mathbf{D}^{(\mu)}(g))^\dagger = (\mathbf{D}^{(\mu)}(g))^{-1} = \mathbf{D}^{(\mu)}(g^{-1})$. Or, when we write explicitly the matrix indices,

$$((\mathbf{D}^{(\mu)}(g))^\dagger)_{ij} = (D^{(\mu)}_{ji}(g))^* = D^{(\mu)}_{ij}(g^{-1})\ . \tag{4.4}$$

4.2 The group algebra

Definition 4.3 *The* "group algebra" $\mathcal{A}(G)$ *of the group* G *over the complex numbers* $\mathbb{C}$ *is a complex vector space with the group elements as basis vectors, and with multiplication defined by the group product. That is,* $\mathcal{A}(G)$ *consists of all linear combinations*

$$a = \sum_{g\in G} a(g)\, g \tag{4.5}$$

with complex coefficients $a(g)$*.*

The "natural scalar product" $(\cdot,\cdot)$ *on the group algebra* $\mathcal{A}(G)$ *is defined such that the natural basis vectors (the group elements) are orthonormal,*

$$(g,h) = \delta_{g,h} \qquad \forall\ g,h \in G\ . \tag{4.6}$$

The vector space dimension of the group algebra $\mathcal{A}(G)$ is the order $|G| = N$ of the group.

It is easily verified that the group product turns $\mathcal{A}(G)$ into an associative linear algebra, with a multiplicative unit equal to the unit element $e \in G$. Explicitly, the product of two elements $a, b \in \mathcal{A}(G)$ is

$$ab = \sum_{g\in G}\sum_{h\in G} a(g)\, b(h)\, gh = \sum_{f\in G}\left(\sum_{g\in G} a(g)\, b(g^{-1}f)\right) f = \sum_{f\in G}\left(\sum_{h\in G} a(fh^{-1})\, b(h)\right) f \ , \quad (4.7)$$

where $f = gh$, $g = fh^{-1}$, $h = g^{-1}f$.

Alternatively, the element $a \in \mathcal{A}(G)$ may be regarded as a complex valued function on G, $a : g \mapsto a(g) \in \mathbb{C}$. The set of all such function is denoted by $\mathbb{C}^G$, thus $\mathcal{A}(G) = \mathbb{C}^G$. Since by definition

$$ab = \sum_{f\in G} ab(f)\, f \ , \quad (4.8)$$

we see that

$$ab(f) = \sum_{g\in G} a(g)\, b(g^{-1}f) = \sum_{h\in G} a(fh^{-1})\, b(h) \ . \quad (4.9)$$

This kind of product of two functions a and b is called a *convolution product*, and is then denoted by $a * b$. Here we will most often use the convolution notation and write $a * b$ instead of ab, in an attempt to make some formulae more readable.

The scalar product of two general elements $a, b \in \mathcal{A}(G)$ is

$$(a, b) = \sum_{g\in G}\sum_{h\in G} a(g)^*\, b(h)\ (g, h) = \sum_{g\in G} a(g)^*\, b(g) \ . \quad (4.10)$$

This kind of scalar product of functions is called an L^2 scalar product.

4.3 The regular representations

The left and right regular realizations L and R, by which each $g \in G$ acts as permutations L_g and R_g of the group elements,

$$L_g(h) = gh \ , \qquad R_g(h) = hg^{-1} \ , \qquad \forall h \in G \ , \quad (4.11)$$

have another straightforward interpretation as linear representations of G, called the left and right *regular representations*, over the N dimensional vector space $\mathcal{A}(G)$. These are both cyclic representations, in fact, every $h \in G$ is a cyclic vector for the representation L as well as for R.

In order to distinguish between L_g as a permutation over G and as a linear transformation over $\mathcal{A}(G)$, we may write $L(g)$ for the latter. If

$$a = \sum_{h\in G} a(h)\, h \ , \quad (4.12)$$

then by definition

$$L(g)\, a = \sum_{h\in G} a(h)\, L(g)\, h = \sum_{h\in G} a(h)\, L_g(h) = \sum_{h\in G} a(h)\, gh = \sum_{f\in G} a(g^{-1}f)\, f\ . \tag{4.13}$$

Alternatively, if we regard $a \in \mathcal{A}(G)$ as a function $a : G \to \mathbb{C}$, we have that

$$[L(g)\, a](f) = a(g^{-1}f) = a(L_g^{-1}(f)) \qquad \forall f \in G\ , \tag{4.14}$$

or,

$$L(g)\, a = aL_g^{-1}\ . \tag{4.15}$$

The left hand side in this equation is a linear transformation $L(g)$ acting on a vector $a \in \mathcal{A}(G)$, whereas the right hand side is the composite of two functions $L_g^{-1} : G \to G$ and $a : G \to \mathbb{C}$.

In a similar way we have that

$$R(g)\, a = \sum_{h\in G} a(h)\, R(g)\, h = \sum_{h\in G} a(h)\, R_g(h) = \sum_{h\in G} a(h)\, hg^{-1} = \sum_{f\in G} a(fg)\, f\ . \tag{4.16}$$

In the alternative notation, with a as a function, we have that

$$[R(g)\, a](f) = a(fg) = a(R_g^{-1}(f))\ , \tag{4.17}$$

or,

$$R(g)\, a = aR_g^{-1}\ . \tag{4.18}$$

Exercise 4.4 *Show that the left and right regular representations are equivalent. (Hint: $R_g(h) = (L_g(h^{-1}))^{-1}\ \forall g, h \in G$.)*

Theorem 4.5 *The left and right regular representations L and R of the group G are both faithful representations.*

Proof. We have to show that $L(g)$ and $L(h)$ are different linear transformations on $\mathcal{A}(G)$ when $g \neq h$. Take for example the unit element $e \in G$, regarded as a vector in $\mathcal{A}(G)$. Then $L(g)\, e = ge = g \neq L(h)\, e = he = h$.

QED

It is easy to verify that the two regular representations are unitary with respect to the natural scalar product on the group algebra. Indeed, it follows from the relation

$$(L(f)\, g, L(f)\, h) = (fg, fh) = \delta_{fg,fh} = \delta_{g,h} = (g, h)\ , \tag{4.19}$$

valid for every $f, g, h \in G$, that $L(f)$ is unitary, thus the left regular representation L is unitary. Similarly, the right regular representation R is unitary.

We may use the natural basis for the group algebra, the group elements, in order to define matrix representations $\mathbf{L}$ and $\mathbf{R}$ equivalent to L and R. The relation

$$L(g)\, h = gh = \sum_{f\in G} L_{f,h}(g)\, f\ , \tag{4.20}$$

which corresponds to eq. (3.4), defines the matrix elements $L_{f,h}(g)$ of the matrix $\mathbf{L}(g)$ representing the linear transformation $L(g)$. We see that the matrix elements are Kronecker δ's,

$$L_{f,h}(g) = \delta_{f,gh} \ . \tag{4.21}$$

Similarly, the relation

$$R(g)\, h = hg^{-1} = \sum_{f \in G} R_{f,h}(g)\, f \tag{4.22}$$

defines the matrix elements of the matrix $\mathbf{R}(g)$,

$$R_{f,h}(g) = \delta_{fg,h} \ . \tag{4.23}$$

4.4 Representation matrices in the group algebra

Let $\mathbf{D}$ be an n dimensional matrix representation of G. This means that $\mathbf{D}(gh) = \mathbf{D}(g)\,\mathbf{D}(h)$ for all $g, h \in G$, or in terms of matrix elements,

$$D_{ik}(gh) = \sum_{j=1}^{n} D_{ij}(g)\, D_{jk}(h) \ . \tag{4.24}$$

Then the matrix elements D_{ij} are complex valued functions on G and may be regarded as elements of the group algebra $\mathcal{A}(G)$,

$$D_{ij} = \sum_{h \in G} D_{ij}(h)\, h \ . \tag{4.25}$$

They span a subspace of $\mathcal{A}(G)$,

$$\mathcal{A}(\mathbf{D}) = \{ \sum_{i,j} a_{ij} D_{ij} \mid a_{ij} \in \mathbb{C} \} \ . \tag{4.26}$$

Theorem 4.6 *Equivalent matrix representations* $\mathbf{D}$ *and* $\widetilde{\mathbf{D}}$ *define the same subspace,* $\mathcal{A}(\mathbf{D}) = \mathcal{A}(\widetilde{\mathbf{D}})$.

Proof. Equivalence of $\mathbf{D}$ and $\widetilde{\mathbf{D}}$ means that there exists an invertible matrix $\mathbf{S}$ such that

$$\widetilde{\mathbf{D}}(g) = \mathbf{S}\,\mathbf{D}(g)\,\mathbf{S}^{-1} \qquad \forall g \in G \ . \tag{4.27}$$

It follows that $\mathcal{A}(\widetilde{\mathbf{D}}) \subset \mathcal{A}(\mathbf{D})$, because

$$\widetilde{D}_{ij} = \sum_{k,m} S_{ik} D_{km} (S^{-1})_{mj} \in \mathcal{A}(\mathbf{D}) \ , \tag{4.28}$$

Similarly, $\mathcal{A}(\mathbf{D}) \subset \mathcal{A}(\widetilde{\mathbf{D}})$.

QED

Theorem 4.7 *To every (finite dimensional) representation ρ of the group G there corresponds a unique subspace of $\mathcal{A}(G)$, $\mathcal{A}(\rho) = \mathcal{A}(\mathbf{D})$, where $\mathbf{D}$ is an arbitrary matrix representation equivalent to ρ.*

If ρ is equivalent to a direct sum

$$\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2 \oplus \ldots \oplus \mathbf{D}_r \;, \tag{4.29}$$

where each $\mathbf{D}_i$ is a matrix representation of dimension n_i, then $\mathcal{A}(\rho)$ is spanned by the ${n_1}^2 + {n_2}^2 + \ldots + {n_r}^2$ matrix elements $[D_i]_{jk}$.

In particular, $\mathcal{A}(\rho)$ is spanned by the matrix elements of the irreducible matrix representations contained in $\mathbf{D}$.

4.5 The irreducible matrix representations

Theorem 4.8 (Completeness) *The matrix elements $D^{(\mu)}_{ij}$ of the irreducible matrix representations span the whole group algebra.*

Proof. By Theorem 4.7, it is enough to find one representation ρ such that $\mathcal{A}(\rho) = \mathcal{A}(G)$, then $\mathcal{A}(G)$ is spanned by the matrix elements of the irreducible matrix representations contained in ρ. One possible choice is $\rho = L$, the left regular representation. In fact, eq. (4.21) gives that

$$L_{f,h} = \sum_{g\in G} L_{f,h}(g)\, g = \sum_{g\in G} \delta_{f,gh}\, g = f h^{-1} \;. \tag{4.30}$$

In order to make the elements $L_{f,h} \in \mathcal{A}(G)$ run through all elements of G, which form the natural basis in $\mathcal{A}(G)$, we may for example fix $h = e$ and let f vary freely.

QED

Assume now that $\mathbf{D}^{(\mu)}$ and $\mathbf{D}^{(\nu)}$ are unitary irreducible representations, inequivalent if $\mu \neq \nu$. We regard the matrix elements $D^{(\mu)}_{ij}$ and $D^{(\nu)}_{kl}$ as elements of the group algebra, and compute their group algebra product,

$$D^{(\mu)}_{ij} * D^{(\nu)}_{kl} = \sum_{f\in G}\sum_{g\in G} D^{(\mu)}_{ij}(g)\, D^{(\nu)}_{kl}(g^{-1}f)\, f = \sum_{f\in G}\sum_{h\in G} D^{(\mu)}_{ij}(fh^{-1})\, D^{(\nu)}_{kl}(h)\, f \;. \tag{4.31}$$

Because $\mathbf{D}^{(\nu)}$ is a representation, we have that

$$D^{(\nu)}_{kl}(g^{-1}f) = \sum_{p} D^{(\nu)}_{kp}(g^{-1})\, D^{(\nu)}_{pl}(f) \;, \tag{4.32}$$

and consequently,

$$D^{(\mu)}_{ij} * D^{(\nu)}_{kl} = \sum_{p}\sum_{g\in G} D^{(\mu)}_{ij}(g)\, D^{(\nu)}_{kp}(g^{-1}) \sum_{f\in G} D^{(\nu)}_{pl}(f)\, f \;. \tag{4.33}$$

Similarly, because $\mathbf{D}^{(\mu)}$ is a representation, we have that

$$D^{(\mu)}_{ij}(fh^{-1}) = \sum_{q} D^{(\mu)}_{iq}(f)\, D^{(\mu)}_{qj}(h^{-1}) \;, \tag{4.34}$$

and,

$$D_{ij}^{(\mu)} * D_{kl}^{(\nu)} = \sum_q \sum_{h\in G} D_{qj}^{(\mu)}(h^{-1})\, D_{kl}^{(\nu)}(h) \sum_{f\in G} D_{iq}^{(\mu)}(f)\, f \ . \tag{4.35}$$

By substituting $g \leftrightarrow h^{-1}$ we see that

$$\sum_{g\in G} D_{ij}^{(\mu)}(g)\, D_{kl}^{(\nu)}(g^{-1}) = \sum_{h\in G} D_{ij}^{(\mu)}(h^{-1})\, D_{kl}^{(\nu)}(h) \ . \tag{4.36}$$

Since we assumed the representations $\mathbf{D}^{(\mu)}$ and $\mathbf{D}^{(\nu)}$ to be unitary, this quantity is just the scalar product of the matrix elements as vectors in the group algebra. Let us introduce the notation

$$C_{ijkl}^{(\mu,\nu)} = \sum_{h\in G} D_{ij}^{(\mu)}(h^{-1})\, D_{kl}^{(\nu)}(h) = \sum_{h\in G} (D_{ji}^{(\mu)}(h))^*\, D_{kl}^{(\nu)}(h) = (D_{ji}^{(\mu)}, D_{kl}^{(\nu)}) \ . \tag{4.37}$$

What we have shown is that

$$D_{ij}^{(\mu)} * D_{kl}^{(\nu)} = \sum_p C_{ijkp}^{(\mu,\nu)}\, D_{pl}^{(\nu)} = \sum_q C_{qjkl}^{(\mu,\nu)}\, D_{iq}^{(\mu)} \ . \tag{4.38}$$

Here $C_{ijkl}^{(\mu,\nu)} \in \mathbb{C}$, whereas $D_{ij}^{(\mu)} \in \mathcal{A}(G)$ and $D_{kl}^{(\nu)} \in \mathcal{A}(G)$. Thus, the second equality in the above equation may be written as

$$\sum_p C_{ijkp}^{(\mu,\nu)}\, D_{pl}^{(\nu)}(f) = \sum_q D_{iq}^{(\mu)}(f)\, C_{qjkl}^{(\mu,\nu)} \qquad \forall f \in G \ . \tag{4.39}$$

Since we assumed the representations $\mathbf{D}^{(\mu)}$ and $\mathbf{D}^{(\nu)}$ to be irreducible, we may now apply Schur's lemma. We first conclude that if $\mu \neq \nu$, so that the two irreducible representations are inequivalent, we must have $C_{ijkl}^{(\mu,\nu)} = 0$. If $\mu = \nu$, then the dependence of $C_{ijkl}^{(\mu,\nu)}$ on the two indices i and l must be the Kronecker delta δ_{il}. Thus we may write

$$C_{ijkl}^{(\mu,\nu)} = B_{jk}^{(\mu)}\, \delta_{\mu\nu}\, \delta_{il} \ . \tag{4.40}$$

The coefficients $B_{jk}^{(\mu)}$ are so far unknown, but are easily calculated. When we set $\mu = \nu$ and $i = l$, and sum over i, we get the equation

$$\sum_i C_{ijki}^{(\mu,\mu)} = B_{jk}^{(\mu)}\, n_\mu \ , \tag{4.41}$$

where n_μ is the dimension of the irreducible representation $\mathbf{D}^{(\mu)}$. The left hand side of this equation is

$$\begin{aligned} \sum_i C_{ijki}^{(\mu,\mu)} &= \sum_i \sum_{h\in G} D_{ij}^{(\mu)}(h^{-1})\, D_{ki}^{(\mu)}(h) = \sum_{h\in G} \sum_i D_{ki}^{(\mu)}(h)\, D_{ij}^{(\mu)}(h^{-1}) \\ &= \sum_{h\in G} D_{kj}^{(\mu)}(hh^{-1}) = \sum_{h\in G} D_{kj}^{(\mu)}(e) = \sum_{h\in G} \delta_{kj} = N\, \delta_{kj} \ . \end{aligned} \tag{4.42}$$

Hence,

$$B_{jk}^{(\mu)} = \frac{N}{n_\mu}\, \delta_{jk} \ . \tag{4.43}$$

To summarize, by a somewhat lengthy computation we have obtained two important results. One is the group algebra product of the matrix elements,

$$D^{(\mu)}_{ij} * D^{(\nu)}_{kl} = \frac{N}{n_\mu}\, \delta_{\mu\nu}\, \delta_{jk}\, D^{(\mu)}_{il} \ . \tag{4.44}$$

And as a byproduct we have proved an orthogonality relation.

Theorem 4.9 (Orthogonality) *The natural scalar products in the group algebra between matrix elements of the irreducible unitary representations are*

$$(D^{(\mu)}_{ij}, D^{(\nu)}_{kl}) = \frac{N}{n_\mu}\, \delta_{\mu\nu}\, \delta_{ik}\, \delta_{jl} \ . \tag{4.45}$$

It follows that these matrix elements, as vectors in the group algebra, are nonzero and linearly independent. Since they span the whole group algebra, they form a basis.

We will show at the end of this chapter that the ratio N/n_μ is always an integer. In other words, the dimension of an irreducible representation is always a divisor of the order of the group.

The scalar product of one matrix element with itself is

$$(D^{(\mu)}_{ij}, D^{(\mu)}_{ij}) = \sum_{g\in G} |D^{(\mu)}_{ij}(g)|^2 = \frac{N}{n_\mu} \ . \tag{4.46}$$

This shows that whatever matrix element we pick in an irreducible representation, there is always at least one group element for which it is nonzero.

Any two bases in a finite dimensional vector space must have the same number of basis vectors. Thus, the finite group G can have only a finite number M of inequivalent irreducible representations, and by counting basis vectors we find a very useful relation between the order of the group, $|G| = N$, and the dimensions of the irreducible representations,

$$N = \sum_{\mu=1}^{M} {n_\mu}^2 \ . \tag{4.47}$$

We will prove below that the number of inequivalent irreducible representations, M, is equal to the number of conjugation classes, K.

The group algebra as a direct sum of matrix algebras

For an element $a = \sum_g a(g)\, g \in \mathcal{A}(G)$ we now define

$$a^{(\mu)}_{ij} = (D^{(\mu)}_{ij}, a) = \sum_g (D^{(\mu)}_{ij}(g))^*\, a(g) = \sum_g D^{(\mu)}_{ji}(g^{-1})\, a(g) \ . \tag{4.48}$$

Then the expansion of a in the orthogonal basis of the irreducible matrix elements is

$$a = \frac{1}{N} \sum_{\mu=1}^{M} \sum_{i=1}^{n_\mu} \sum_{j=1}^{n_\mu} n_\mu\, a^{(\mu)}_{ij}\, D^{(\mu)}_{ij} \ . \tag{4.49}$$

The product of a with another element $b \in \mathcal{A}(G)$, expanded as

$$b = \frac{1}{N} \sum_{\nu=1}^{M} \sum_{k=1}^{n_\nu} \sum_{l=1}^{n_\nu} n_\nu \, b_{kl}^{(\nu)} \, D_{kl}^{(\nu)} \,, \tag{4.50}$$

is

$$\begin{aligned} a * b \; &= \; \frac{1}{N^2} \sum_{\mu} \sum_{i} \sum_{j} \sum_{\nu} \sum_{k} \sum_{l} n_\mu \, n_\nu \, a_{ij}^{(\mu)} \, b_{kl}^{(\nu)} \, D_{ij}^{(\mu)} * D_{kl}^{(\nu)} \\ &= \frac{1}{N} \sum_{\mu} \sum_{i} \sum_{l} n_\mu \, c_{il}^{(\mu)} \, D_{il}^{(\mu)} \,. \end{aligned} \tag{4.51}$$

with

$$c_{il}^{(\mu)} = \sum_{j} a_{ij}^{(\mu)} \, b_{jl}^{(\mu)} \,. \tag{4.52}$$

This shows that the group algebra product $a * b = c$ corresponds to M matrix products $\mathbf{A}^{(\mu)}\mathbf{B}^{(\mu)} = \mathbf{C}^{(\mu)}$, where the matrix elements of the matrices $\mathbf{A}^{(\mu)}, \mathbf{B}^{(\mu)}, \mathbf{C}^{(\mu)}$ are the components $a_{ij}^{(\mu)}, b_{ij}^{(\mu)}, c_{ij}^{(\mu)}$ of $a, b, c \in \mathcal{A}(G)$.

Thus, we may picture the general element $a \in \mathcal{A}(G)$ as a block diagonal matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{(2)} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}^{(M)} \end{pmatrix} , \tag{4.53}$$

where each diagonal block $\mathbf{A}^{(\mu)}$ is a general $n_\mu \times n_\mu$ matrix. And the group algebra product is the same as the matrix product of such matrices.

4.6 The centre of the group algebra

An important subalgebra of the group algebra is its centre, or commutant, defined as

$$\mathcal{A}(G)' = \{ \, c \in \mathcal{A}(G) \mid c * a = a * c \quad \forall \, a \in \mathcal{A}(G) \, \} \,. \tag{4.54}$$

The centre is commutative, by definition.

We will use two different approaches for analysing the centre of the group algebra. Let $c \in \mathcal{A}(G)'$. One approach will give c as a linear combination of the conjugation classes, and the other approach will give c as a linear combination of the irreducible characters.

The conjugation classes as a basis

Obviously, an element $c \in \mathcal{A}(G)$ belongs to $\mathcal{A}(G)'$ if and only if it commutes with every $h \in G$.

Recall that the conjugation class of a group element $g \in G$ is the subset

$$C = \{hgh^{-1} \mid h \in G\} \subset G \,. \tag{4.55}$$

Here we introduce the conjugation class C also as an element of the group algebra,

$$C = \sum_{g \in C} g \ . \tag{4.56}$$

Then C commutes with every $h \in G$, that is, $h * C = C * h$, because

$$h * C * h^{-1} = \sum_{g \in C} hgh^{-1} = C \ . \tag{4.57}$$

The effect of the mapping $g \to hgh^{-1}$ is just to permute the elements of C.

Let K be the number of conjugation classes of the group G, and enumerate them as $C_1, C_2, \ldots, C_K$. They are linearly independent vectors in $\mathcal{A}(G)$, and they are orthogonal,

$$(C_i, C_j) = \sum_{g \in C_i} \sum_{h \in C_j} (g, h) = \sum_{g \in C_i} \sum_{h \in C_j} \delta_{g,h} = N_i \, \delta_{ij} \ , \tag{4.58}$$

where $N_i = |C_i|$ is the number of elements of C_i. The unit element $e \in G$ is a conjugation class by itself, and it is natural to define $C_1 = \{e\}$, with $N_1 = 1$.

As we have seen, the conjugation classes belong to the centre $\mathcal{A}(G)'$. Moreover, if

$$c = \sum_{g \in G} c(g) \, g \in \mathcal{A}(G)' \ , \tag{4.59}$$

then c is a linear combination of conjugation classes, because

$$c = \frac{1}{N} \sum_{h \in G} h * c * h^{-1} = \frac{1}{N} \sum_{g \in G} c(g) \sum_{h \in G} hgh^{-1} = \sum_{i=1}^{K} \left(\frac{1}{N_i} \sum_{g \in C_i} c(g) \right) C_i \ . \tag{4.60}$$

The last equality follows because when C_i is the conjugation class of g, then

$$\frac{1}{N} \sum_{h \in G} hgh^{-1} = \frac{1}{N_i} C_i \ . \tag{4.61}$$

This completes the proof of the following theorem.

Theorem 4.10 *The conjugation classes $C_1, C_2, \ldots, C_K$ are basis vectors of the centre $\mathcal{A}(G)'$. They are orthogonal in the natural scalar product,*

$$(C_i, C_j) = N_i \, \delta_{ij} \ . \tag{4.62}$$

The irreducible characters as a basis

The character χ of any linear representation ρ is a class function, $\chi(g) = \chi_i$ for $g \in C_i$. Therefore, when χ is regarded as a member of the group algebra $\mathcal{A}(G)$ it is a linear combination of conjugation classes,

$$\chi = \sum_{g \in G} \chi(g) \, g = \sum_{i=1}^{K} \chi_i \, C_i \ . \tag{4.63}$$

From what we have just learned, it follows that $\chi \in \mathcal{A}(G)'$.

We now introduce the characters of the irreducible representations,

$$\chi^{(\mu)}(g) = \mathrm{Tr}\,\mathbf{D}^{(\mu)}(g) = \sum_{i=1}^{n_\mu} D_{ii}^{(\mu)}(g) \;, \tag{4.64}$$

and treat them as members of the group algebra,

$$\chi^{(\mu)} = \sum_{g\in G} \chi^{(\mu)}(g)\, g = \sum_{i=1}^{n_\mu} D_{ii}^{(\mu)} = \sum_{i=1}^{n_\mu}\sum_{g\in G} D_{ii}^{(\mu)}(g)\, g \;. \tag{4.65}$$

The orthogonality relation (4.45) implies directly that

$$(\chi^{(\mu)}, \chi^{(\nu)}) = \sum_i \sum_k (D_{ii}^{(\mu)}, D_{kk}^{(\nu)}) = \sum_i \sum_k \frac{N}{n_\mu}\, \delta_{\mu\nu}\, \delta_{ik}\, \delta_{ik} = N\, \delta_{\mu\nu} \;. \tag{4.66}$$

Only the identity matrix and multiples of it commute with every matrix in a full matrix algebra. For example, the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.67}$$

commutes with the two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.68}$$

if and only if $b = c = 0$ and $a = d$.

Hence, when we represent $c \in \mathcal{A}(G)'$ by a block diagonal matrix $\mathbf{C}$, like in eq. (4.53),

$$\mathbf{C} = \begin{pmatrix} \mathbf{C}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{(2)} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{C}^{(M)} \end{pmatrix}, \tag{4.69}$$

it follows that the blocks on the diagonal must be multiples of the identity, $\mathbf{C}^{(\mu)} = c^{(\mu)}\mathbf{I}^{(\mu)}$, where $c^{(\mu)} \in \mathbb{C}$ and $\mathbf{I}^{(\mu)}$ is the $n_\mu \times n_\mu$ identity matrix. This means that c is a linear combination of the irreducible characters, in fact,

$$c = \frac{1}{N}\sum_\mu \sum_i \sum_l n_\mu\, c^{(\mu)}\, \delta_{il}\, D_{il}^{(\mu)} = \frac{1}{N}\sum_\mu \sum_i n_\mu\, c^{(\mu)}\, D_{ii}^{(\mu)} = \frac{1}{N}\sum_\mu n_\mu\, c^{(\mu)}\, \chi^{(\mu)} \;. \tag{4.70}$$

We have now established the conjugation classes C_i and the irreducible characters $\chi^{(\mu)}$ as two different orthogonal bases for $\mathcal{A}(G)'$. Since two bases of a finite dimensional vector space must have the same number of basis vectors, it follows that the number of irreducible representations is equal to the number of conjugation classes.

We may write the irreducible characters as linear combinations of the conjugation classes,

$$\chi^{(\mu)} = \sum_{g\in G} \chi^{(\mu)}(g)\, g = \sum_{i=1}^{K} \chi_i^{(\mu)}\, C_i \;. \tag{4.71}$$

The scalar product of $\chi^{(\mu)}$ with C_i is

$$(\chi^{(\mu)}, C_i) = \sum_{g\in C} (\chi^{(\mu)}(g))^* = N_i\, (\chi_i^{(\mu)})^* \; . \tag{4.72}$$

And we may write the conjugation classes are linear combinations of the irreducible characters,

$$C_i = \frac{1}{N}\sum_{\mu=1}^{K} (\chi^{(\mu)}, C_i)\, \chi^{(\mu)} = \frac{N_i}{N}\sum_{\mu=1}^{K} (\chi_i^{(\mu)})^*\, \chi^{(\mu)} \; . \tag{4.73}$$

Let us sum up in a theorem.

Theorem 4.11 *The number of irreducible representations of the finite group G is K, the number of conjugation classes in G.*

The irreducible characters $\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(K)}$ are basis vectors of the centre $\mathcal{A}(G)'$ of the group algebra $\mathcal{A}(G)$. They are orthogonal in the natural scalar product,

$$(\chi^{(\mu)}, \chi^{(\nu)}) = \sum_{g\in G} (\chi^{(\mu)}(g))^*\, \chi^{(\nu)}(g) = \sum_{i=1}^{K} N_i\, (\chi_i^{(\mu)})^*\, \chi_i^{(\nu)} = N\, \delta_{\mu\nu} \; . \tag{4.74}$$

An equivalent orthogonality relation is the following,

$$\sum_{\mu=1}^{K} (\chi_i^{(\mu)})^*\, \chi_j^{(\mu)} = \frac{N}{N_i}\, \delta_{ij} \; . \tag{4.75}$$

The second orthogonality relation may also be called a completeness relation, because it is what we need to derive eq. (4.73) from eq. (4.71).

To see that the relations (4.74) and (4.75) are equivalent, consider the $K \times K$ matrix $\mathbf{M}$ with matrix elements

$$M_{\mu i} = \sqrt{\frac{N_i}{N}}\, \chi_i^{(\mu)} \; . \tag{4.76}$$

Eq. (4.74) says that $\mathbf{M}\mathbf{M}^\dagger = \mathbf{I}$, whereas eq. (4.75) says that $\mathbf{M}^\dagger\mathbf{M} = \mathbf{I}$. In the case of finite dimensional square matrices any one of the two equations $\mathbf{M}\mathbf{M}^\dagger = \mathbf{I}$ and $\mathbf{M}^\dagger\mathbf{M} = \mathbf{I}$ implies that $\mathbf{M}$ is invertible, with $\mathbf{M}^{-1} = \mathbf{M}^\dagger$, and then the other equation follows.

4.7 More about characters

If a linear representation ρ is a direct sum,

$$\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n \; , \tag{4.77}$$

then its character χ is a sum,

$$\chi = \chi_1 + \chi_2 + \cdots + \chi_n \; . \tag{4.78}$$

Since ρ can always be decomposed as a direct sum of irreducible representations, its character is a sum over the irreducible characters,

$$\chi = \sum_{\mu} m_\mu \, \chi^{(\mu)} \ , \tag{4.79}$$

where $m_\mu = 0, 1, 2, \ldots$ is the multiplicity with which the irreducible representation μ occurs in the direct sum. Given the character χ, we may use the orthogonality relation of the irreducible characters to compute this multiplicity,

$$m_\mu = \frac{1}{N} \sum_{g \in G} (\chi^{(\mu)}(g))^* \, \chi(g) = \frac{1}{N} \sum_{i=1}^{K} N_i \, (\chi_i^{(\mu)})^* \, \chi_i \ . \tag{4.80}$$

The irreducible characters as projection operators

The group algebra product of a character and a matrix element is

$$\chi^{(\mu)} * D_{kl}^{(\nu)} = \sum_i D_{ii}^{(\mu)} * D_{kl}^{(\nu)} = \sum_i \frac{N}{n_\mu} \, \delta_{\mu\nu} \, \delta_{ik} \, D_{il}^{(\mu)} = \frac{N}{n_\mu} \, \delta_{\mu\nu} \, D_{kl}^{(\nu)} \ . \tag{4.81}$$

The product in the opposite order is the same,

$$D_{kl}^{(\nu)} * \chi^{(\mu)} = \sum_i D_{kl}^{(\nu)} * D_{ii}^{(\mu)} = \sum_i \frac{N}{n_\nu} \, \delta_{\nu\mu} \, \delta_{li} \, D_{ki}^{(\nu)} = \frac{N}{n_\mu} \, \delta_{\mu\nu} \, D_{kl}^{(\nu)} \ . \tag{4.82}$$

Thus, the result of multiplying any $a \in \mathcal{A}(G)$ by $\chi^{(\mu)}$, from the left or from the right, is to project out the component of a lying in the subspace $\mathcal{A}(\mathbf{D}^{(\mu)})$. Defining

$$e_\mu = \frac{n_\mu}{N} \, \chi^{(\mu)} \ , \tag{4.83}$$

and using the expansion introduced in eq. (4.49), we have the identity

$$\sum_\mu e_\mu * a = \frac{1}{N} \sum_\mu \sum_i \sum_j n_\mu \, a_{ij}^{(\mu)} \, D_{ij}^{(\mu)} = a \ . \tag{4.84}$$

The special choice $a = e$, the unit element of the group G and of the group algebra $\mathcal{A}(G)$, gives that

$$\sum_\mu e_\mu = \sum_\mu e_\mu * e = e \ . \tag{4.85}$$

The algebra product of two irreducible characters is

$$\chi^{(\mu)} * \chi^{(\nu)} = \frac{N}{n_\mu} \, \delta_{\mu\nu} \, \chi^{(\mu)} \ . \tag{4.86}$$

Thus, the elements e_μ of the group algebra, defined above, are projections with the property that

$$e_\mu * e_\nu = \delta_{\mu\nu} \, e_\mu \ . \tag{4.87}$$

Eq. (4.86) is an equality between N dimensional vectors. The component of the vector equation relative to the basis vector e is

$$\sum_{g\in G} \chi^{(\mu)}(g^{-1})\,\chi^{(\nu)}(g) = \frac{N}{n_\mu}\,\delta_{\mu\nu}\,\chi^{(\mu)}(e)\ . \tag{4.88}$$

The unitarity of the representation $\mathbf{D}^{(\mu)}$ implies that

$$\chi^{(\mu)}(g^{-1}) = \mathrm{Tr}\,\mathbf{D}^{(\mu)}(g^{-1}) = \mathrm{Tr}(\mathbf{D}^{(\mu)}(g))^\dagger = (\chi^{(\mu)}(g))^*\ . \tag{4.89}$$

Note that it is a more or less arbitrary convention when we choose the matrix representation $\mathbf{D}^{(\mu)}$ to be unitary, but the relation $\chi^{(\mu)}(g^{-1}) = (\chi^{(\mu)}(g))^*$ does not depend on any particular convention, because the character is invariant under similarity transformations. Since

$$\chi^{(\mu)}(e) = \mathrm{Tr}\,\mathbf{D}^{(\mu)}(e) = \mathrm{Tr}\,\mathbf{I} = n_\mu\ , \tag{4.90}$$

we arrive once more at the orthogonality relation of the irreducible characters,

$$(\chi^{(\mu)},\chi^{(\nu)}) = \sum_{g\in G}(\chi^{(\mu)}(g))^*\,\chi^{(\nu)}(g) = N\,\delta_{\mu\nu}\ . \tag{4.91}$$

Number theoretical properties of characters

Every element g of a finite group G has a finite order $n \geq 1$, such that $g^n = e$, and $g^k \neq e$ if $1 \leq k < n$. In any matrix representation $\mathbf{D}$ we must have that

$$(\mathbf{D}(g))^n = \mathbf{I}\ . \tag{4.92}$$

An eigenvalue λ of $\mathbf{D}(g)$ is then an n-th root of unity, it is one of the roots

$$\lambda_k = \mathrm{e}^{\mathrm{i}\frac{2k\pi}{n}}\ , \qquad k = 0, 1, \ldots, n-1 \tag{4.93}$$

of the characteristic equation

$$\lambda^n = 1\ . \tag{4.94}$$

The matrix $\mathbf{D}(g)$ can be completely diagonalized, because it is either unitary or the similarity transform of a unitary matrix, and every unitary matrix can be diagonalized. By the way, the fact that $\mathbf{D}(g)$ is diagonalizable also follows from Theorem B.38, because the polynomial equation

$$(\mathbf{D}(g))^n - \mathbf{I} = \prod_{k=0}^{n-1}(\mathbf{D}(g) - \lambda_k\mathbf{I}) = 0 \tag{4.95}$$

has n different complex roots λ_k.

Anyway, the character value $\chi(g) = \mathrm{Tr}\,\mathbf{D}(g)$ is the sum of the eigenvalues of $\mathbf{D}(g)$, which are n-th roots of unity. Therefore it is an *algebraic integer*, as defined in Appendix A.

Theorem 4.12 *A character value $\chi(g)$ for an element g of a finite group G is an algebraic integer. In particular, if $\chi(g)$ is a rational number, it is an integer.*

The ratio N/n_μ of the group order N and the dimension n_μ of an irreducible representation is also an algebraic integer. Since it is a rational number, it is an integer.

That N/n_μ is an algebraic integer, and hence an integer, is a rather deep result. The proof goes as follows. We read eq. (4.86) with $\mu = \nu$ as an eigenvalue equation for the irreducible character $\chi^{(\mu)}$ as a linear operator,

$$\chi^{(\mu)} * \chi^{(\mu)} = \frac{N}{n_\mu} \chi^{(\mu)} . \tag{4.96}$$

In words: the operator $\chi^{(\mu)}$ has $\chi^{(\mu)}$ as an eigenvector with eigenvalue N/n_μ.

Now use the conjugation classes $C_1, C_2, \ldots, C_K$ as a basis for the centre $\mathcal{A}(G)'$. The multiplication table of the conjugation classes is given by a set of constants c_{ij}^k that we may call *structure constants* of the commutative algebra $\mathcal{A}(G)'$,

$$C_i * C_j = \sum_k c_{ij}^k \, C_k . \tag{4.97}$$

Every coefficient c_{ij}^k is either zero or a positive integer. To prove this, use the definition

$$C_i * C_j = \sum_{g \in C_i} \sum_{h \in C_j} gh . \tag{4.98}$$

Given one element $g_1 \in C_k$, every element of C_k is a conjugate $g_1' = f g_1 f^{-1}$ for some $f \in G$. If $g_1 = gh$ with $g \in C_i$, $h \in C_j$, then $g_1' = g'h'$ with $g' = fgf^{-1} \in C_i$, $h' = fhf^{-1} \in C_j$. This shows that the two equations $g_1 = gh$ and $g_1' = g'h'$ have the same number of solutions. The coefficient c_{ij}^k is just that number of solutions.

We choose to read eq. (4.97) in the following way. We regard C_i as a linear transformation of the basis vector C_j into a linear combination of basis vectors C_k with $k = 1, 2, \ldots, K$. In analogy with eq. (3.4), this gives us a matrix representation of the linear algebra of conjugation classes, $C_i \to \mathbf{C}_i$, such that the matrix elements of $\mathbf{C}_i$ are

$$(\mathbf{C}_i)_{kj} = c_{ij}^k . \tag{4.99}$$

Since the character $\chi^{(\mu)}$ is a linear combination of the conjugation classes,

$$\chi^{(\mu)} = \sum_{i=1}^{K} \chi_i^{(\mu)} \, C_i , \tag{4.100}$$

where the character values $\chi_i^{(\mu)} \in \mathbb{C}$ are algebraic integers, we have a representation of $\chi^{(\mu)}$ as a matrix

$$\mathbf{A}^{(\mu)} = \sum_{i=1}^{K} \chi_i^{(\mu)} \, \mathbf{C}_i , \tag{4.101}$$

with matrix elements that are algebraic integers. The characteristic polynomial of $\mathbf{A}^{(\mu)}$,

$$\det(\lambda \mathbf{I} - \mathbf{A}^{(\mu)}) = \lambda^{n_\mu} + a_{n_\mu - 1} \lambda^{n_\mu - 1} + \cdots a_1 \lambda + a_0 , \tag{4.102}$$

has coefficients a_i that are algebraic integers. Hence, every eigenvalue λ, including $\lambda = N/n_\mu$, is an algebraic integer.