Contents

Chapter 1

Representations of the symmetric group S_n

1.1 The one dimensional representations

We know already two irreducible representations of the symmetric group S_n, both one dimensional. Every group has the trivial representation $p \mapsto 1$, we may call this the symmetric representation of S_n. In the case of S_n there is also the sign representation $p \mapsto \operatorname{sgn}(p) = \pm 1$, which we may call the antisymmetric representation.

To see that these are the only one dimensional representations, recall that every permutation $p \in S_n$ may be written as a product of transpositions. A transposition $t = (ij)$ interchanges two numbers i and j, and is its own inverse, $t^2 = e$. A one dimensional representation ρ of S_n must represent the unit element $e \in S_n$ by the number 1. Since

$$(\rho(t))^2 = \rho(t^2) = \rho(e) = 1 \;, \tag{1.1}$$

we must have either $\rho(t) = 1$ or $\rho(t) = -1$. Any transposition $u = (kl)$ is a conjugate of $t = (ij)$, in fact $u = qtq^{-1}$ with either $q(i) = k$, $q(j) = l$ or $q(i) = l$, $q(j) = k$. Since a one dimensional representation is commutative, it follows that

$$\rho(u) = \rho(qtq^{-1}) = \rho(q)\rho(t)\rho(q^{-1}) = \rho(t)\rho(q)\rho(q^{-1}) = \rho(t)\rho(qq^{-1}) = \rho(t)\rho(e) = \rho(t) \;. \tag{1.2}$$

Either we have $\rho(u) = 1$ for every transposition u, this is the symmetric representation, or else we have $\rho(u) = -1$ for every transposition u, and this is the antisymmetric representation.

Define the symmetrizer

$$S = \frac{1}{n!} \sum_{p \in S_n} p \tag{1.3}$$

and the antisymmetrizer

$$A = \frac{1}{n!} \sum_{p \in S_n} \operatorname{sgn}(p)\, p \tag{1.4}$$

as members of the group algebra $\mathcal{A}(S_n)$. For an arbitrary permutation $q \in S_n$ we have that

$$qS = Sq = S \;, \qquad qA = Aq = \operatorname{sgn}(q)\, A \;. \tag{1.5}$$

The proof is easy,

$$qS = \frac{1}{n!} \sum_{p \in S_n} qp = \frac{1}{n!} \sum_{r \in S_n} r = S \; , \tag{1.6}$$

where we define $r = qp$, and similarly,

$$\begin{aligned} qA &= \frac{1}{n!} \sum_{p \in S_n} \operatorname{sgn}(p)\, qp = \frac{1}{n!} \sum_{p \in S_n} \operatorname{sgn}(q)\, \operatorname{sgn}(qp)\, qp \\ &= \frac{\operatorname{sgn}(q)}{n!} \sum_{r \in S_n} \operatorname{sgn}(r)\, r = \operatorname{sgn}(q)\, A \; . \end{aligned} \tag{1.7}$$

It follows that S and A are projections, $S^2 = S$ and $A^2 = A$. In fact,

$$S^2 = \frac{1}{n!} \sum_{p \in S_n} pS = \frac{1}{n!} \sum_{p \in S_n} S = S \; , \tag{1.8}$$

and

$$A^2 = \frac{1}{n!} \sum_{p \in S_n} \operatorname{sgn}(p)\, pA = \frac{1}{n!} \sum_{p \in S_n} (\operatorname{sgn}(p))^2\, A = \frac{1}{n!} \sum_{p \in S_n} A = A \; . \tag{1.9}$$

They project out the symmetric and the antisymmetric representation, respectively. Note also that $SA = AS = 0$, in fact,

$$SA = \frac{1}{n!} \sum_{p \in S_n} pA = \frac{1}{n!} \sum_{p \in S_n} \operatorname{sgn}(p)\, A = 0 \; . \tag{1.10}$$

1.2 Irreducible representations from Young diagrams

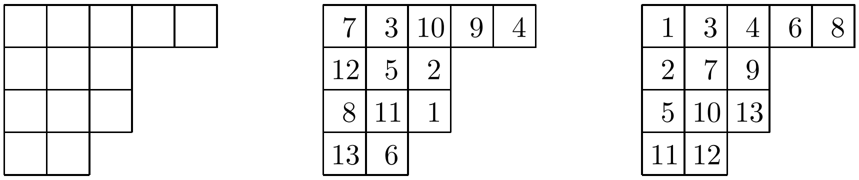
We have learnt in the general theory of complex linear representations of finite groups that the number of irreducible representations is the same as the number of conjugation classes. Although this means that there exists a one to one correspondence between conjugation classes and irreducible representations, there is in general no canonical way to associate one particular conjugation class with one particular irreducible representation. In the special case of the symmetric group S_n it is possible to make the one to one correspondence quite explicit, because we use partitions of n to label both the conjugation classes and the irreducible representations.

A partition of the positive integer n is a sequence of integers

$$L_1 \geq L_2 \geq \ldots \geq L_k > 0 \qquad \text{with} \qquad L_1 + L_2 + \cdots + L_k = n \; . \tag{1.11}$$

We may picture the partition by a *Young diagram*, an arrangement of n empty boxes into rows and columns, with L_i boxes in row number i. The row lengths decrease from top to bottom, and the column heights decrease from left to right. To every Young diagram there is a *conjugate* diagram, which is reflected about the diagonal so that rows become columns and columns become rows.

A *Young tableau* is a Young diagram where the boxes have been filled in with the numbers $1, 2, \ldots, n$, in some order. A *standard Young tableau* is a Young tableau where the numbers are in increasing order from left to right in every row and from top to bottom in every column. Figure 1.1 shows as an example the Young diagram with $n = 13 = 5+3+3+2$, and two corresponding tableaux, one non-standard and one standard. The conjugate partition is $13 = 4+4+3+1+1$.



7	3	10	9	4
12	5	2		
8	11	1		
13	6			

1	3	4	6	8
2	7	9		
5	10	13		
11	12			

Figure 1.1: A Young diagram, a Young tableau, and a standard Young tableau.

It is useful to order the Young tableaux corresponding to one Young diagram, for example in what we choose to call here the standard order. We read the numbers in a tableau from left to right in each row, from the top row down to the bottom row. Given two different tableaux $t_a \neq t_b$, we define that $t_a < t_b$ if the number n appears later in t_a than in t_b. If the two tableaux have n in the same place, then we look for the number $n-1$ and define that $t_a < t_b$ if $n-1$ appears later in t_a than in t_b. And so on down to the number 2, if necessary.

A linear representation from a Young diagram

One way to define the irreducible representation of S_n associated with a Young diagram is the following, which Young introduced. The idea is to use the standard Young tableux of the Young diagram as orthonormal basis vectors.

We have to define how a permutation $p \in S_n$ acts on these basis vectors. Since the special transpositions $T_i = (i, i+1)$ with $i = 1, 2, \ldots, n-1$ generate the whole group S_n, and the multiplication table of the group follows from the basic relations

$$\begin{aligned} (T_i)^2 &= I \; , \\ T_i \, T_j &= T_j \, T_i \qquad \text{if} \qquad |i-j| \geq 2 \; , \\ T_{i+1} \, T_i \, T_{i+1} &= T_i \, T_{i+1} \, T_i \; , \end{aligned} \tag{1.12}$$

it is enough to represent these transpositions and their basic relations.

To be concrete, let us illustrate with the example of $n = 13$, with the Young diagram in Figure 1.1, and the two transpositions $T_5 = (5,6)$ and $T_6 = (6,7)$, which have to satisfy the relation $T_6 T_5 T_6 = T_5 T_6 T_5$.

Consider first T_5. The simplest cases are when a standard tableau t_a contains the numbers 5 and 6 in the same row, or when another standard tableau t_b contains these two numbers in the same column. Then we define

$$T_5 \, t_a = t_a \; , \qquad T_5 \, t_b = -t_b \; . \tag{1.13}$$

The general principle is: symmetry along a row and antisymmetry along a column.

When the numbers 5 and 6 are in different rows and different columns in a standard Young tableau, for example the one shown in Figure 1.1, then the interchange of these two numbers produces a different standard tableau. Let t_c and t_d be two standard tableaux, in the standard order, differing only by an interchange of 5 and 6, as shown in Figure 1.2. Then we define

$$T_5\, t_c = -rt_c + qt_d \; , \qquad T_5\, t_d = qt_c + rt_d \; , \tag{1.14}$$

with $q = \sqrt{1-r^2}$. We take $1/r$ to be the distance between the numbers 5 and 6 in the tableaux, defined as the number of horizontal steps plus the number of vertical steps. In the example of Figure 1.2 we have $1/r = 5$.

This example shows the general definition. But we should argue why the construction works, that it defines a representation, and that the representation is irreducible.

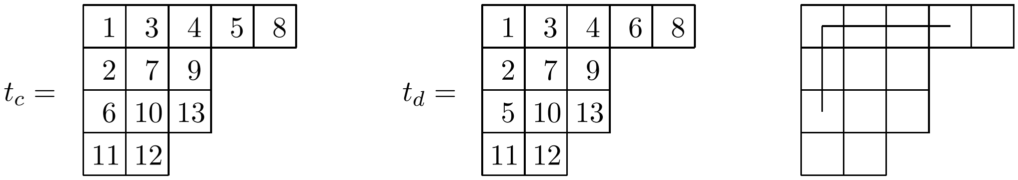


Figure 1.2: Two standard tableaux with 5 and 6 interchanged. How to measure distance.

1.3 Checking the basic relations

Let us continue with the same example. We easily verify that the relation $T_5^2 = I$ follows from the equations (1.13) and (1.14). Equally obvious is the fact that T_i and T_j operate independently whenever $|i-j| \geq 2$, so that they commute, $T_iT_j = T_jT_i$. The non-trivial relation is the third one, so let us investigate how to prove the special case $T_6T_5T_6 = T_5T_6T_5$ in our example.

The transpositions T_5 and T_6 act only on the numbers $5, 6, 7$. The two trivial cases are when a standard tableau t_e contains $5, 6, 7$ in one single row, and when another standard tableau t_f contains $5, 6, 7$ in one single column. Then we have that $T_5\, t_e = T_6\, t_e = t_e$ and $T_5\, t_f = T_6\, t_f = -t_f$, hence

$$T_6T_5T_6\, t_e = T_5T_6T_5\, t_e = t_e \; , \qquad T_6T_5T_6\, t_f = T_5T_6T_5\, t_f = -t_f \; . \tag{1.15}$$

Figure 1.3 is an example of the next simplest case, with two standard tableaux t_g and t_h that get mixed by the transposition T_6. As usual, the standard ordering of the tableaux is significant. The above rules give that

$$T_5\, t_g = t_g \; , \qquad T_5\, t_h = -t_h \; , \tag{1.16}$$

and

$$T_6\, t_g = -rt_g + qt_h \; , \qquad T_6\, t_h = qt_g + rt_h \; , \tag{1.17}$$

$$t_g = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 6 & 8 \\ \hline 2 & 4 & 7 \\ \cline{1-3} 9 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array} \qquad t_h = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 7 & 8 \\ \hline 2 & 4 & 6 \\ \cline{1-3} 9 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array}$$

Figure 1.3: Two standard tableaux in standard order.

with $q = \sqrt{1-r^2}$. According to the rules we should take $1/r = 2$. The two basis vectors t_g and t_h span a two dimensional subspace in which T_5 and T_6 are represented by the 2×2 matrices

$$\mathbf{T}_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{T}_6 = \begin{pmatrix} -r & q \\ q & r \end{pmatrix}. \tag{1.18}$$

With this matrix representation it is easy to check that the relation $\mathbf{T}_6\mathbf{T}_5\mathbf{T}_6 = \mathbf{T}_5\mathbf{T}_6\mathbf{T}_5$ holds if and only if $r = 1/2$.

$$t_i = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 6 & 8 \\ \hline 2 & 4 & 9 \\ \cline{1-3} 7 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array} \qquad t_j = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 7 & 8 \\ \hline 2 & 4 & 9 \\ \cline{1-3} 6 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array} \qquad t_k = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 6 & 7 & 8 \\ \hline 2 & 4 & 9 \\ \cline{1-3} 5 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array}$$

Figure 1.4: Three standard tableaux in standard order.

The third simplest case is the one exemplified in Figure 1.4. These basis vectors define a three dimensional subspace with the matrix representation

$$\mathbf{T}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -r_1 & q_1 \\ 0 & q_1 & r_1 \end{pmatrix}, \qquad \mathbf{T}_6 = \begin{pmatrix} -r_2 & q_2 & 0 \\ q_2 & r_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.19}$$

where $q_1 = \sqrt{1-r_1^2}$ and $q_2 = \sqrt{1-r_2^2}$. To follow the rules we should take $1/r_1 = 4$ and $1/r_2 = 5$. A somewhat long but straightforward calculation using these 3×3 matrices shows that the relation $\mathbf{T}_6\mathbf{T}_5\mathbf{T}_6 = \mathbf{T}_5\mathbf{T}_6\mathbf{T}_5$ holds if and only if

$$\frac{1}{r_2} = \frac{1}{r_1} + 1\,. \tag{1.20}$$

A similar case is exemplified in Figure 1.5. These basis vectors define a three dimensional subspace with the matrix representation

$$\mathbf{T}_5 = \begin{pmatrix} -r_3 & q_3 & 0 \\ q_3 & r_3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \mathbf{T}_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -r_4 & q_4 \\ 0 & q_4 & r_4 \end{pmatrix}, \tag{1.21}$$

$$t_l = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 8 & 9 \\ \cline{1-3} 7 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array} \qquad t_m = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 6 \\ \hline 5 & 8 & 9 \\ \cline{1-3} 7 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array} \qquad t_n = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 7 \\ \hline 5 & 8 & 9 \\ \cline{1-3} 6 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array}$$

Figure 1.5: Three standard tableaux in standard order.

where $q_3 = \sqrt{1 - r_3^2}$ and $q_4 = \sqrt{1 - r_4^2}$. Our rules prescribe $1/r_3 = 5$ and $1/r_4 = 6$. Using these 3×3 matrices we find that $\mathbf{T}_6\mathbf{T}_5\mathbf{T}_6 = \mathbf{T}_5\mathbf{T}_6\mathbf{T}_5$ if and only if

$$\frac{1}{r_4} = \frac{1}{r_3} + 1 \, . \tag{1.22}$$

$$t_o = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 8 \\ \hline 2 & 6 & 9 \\ \cline{1-3} 7 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array} \qquad t_p = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 6 & 8 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 7 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array} \qquad t_q = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 8 \\ \hline 2 & 7 & 9 \\ \cline{1-3} 6 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array}$$

$$t_r = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 6 & 8 \\ \hline 2 & 7 & 9 \\ \cline{1-3} 5 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array} \qquad t_s = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 7 & 8 \\ \hline 2 & 5 & 9 \\ \cline{1-3} 6 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array} \qquad t_t = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 7 & 8 \\ \hline 2 & 6 & 9 \\ \cline{1-3} 5 & 10 & 13 \\ \cline{1-3} 11 & 12 \\ \cline{1-2} \end{array}$$

Figure 1.6: Six standard tableaux in standard order.

The final case is exemplified in Figure 1.6 and involves six basis vectors that get mixed by the transpositions T_5 and T_6. In this subspace we have the matrix representation

$$\mathbf{T}_5 = \begin{pmatrix} -r_5 & q_5 & 0 & 0 & 0 & 0 \\ q_5 & r_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r_7 & q_7 & 0 & 0 \\ 0 & 0 & q_7 & r_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r_6 & q_6 \\ 0 & 0 & 0 & 0 & q_6 & r_6 \end{pmatrix} , \quad \mathbf{T}_6 = \begin{pmatrix} -r_6 & 0 & q_6 & 0 & 0 & 0 \\ 0 & -r_7 & 0 & 0 & q_7 & 0 \\ q_6 & 0 & r_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_5 & 0 & q_5 \\ 0 & q_7 & 0 & 0 & r_7 & 0 \\ 0 & 0 & 0 & q_5 & 0 & r_5 \end{pmatrix} , \tag{1.23}$$

with $q_i = \sqrt{1 - r_i^2}$ for $i = 5, 6, 7$. By the rules we should take $1/r_5 = 3$, $1/r_6 = 2$, and

$1/r_7 = 5$. We find that $\mathbf{T}_6\mathbf{T}_5\mathbf{T}_6 = \mathbf{T}_5\mathbf{T}_6\mathbf{T}_5$ if and only if

$$\frac{1}{r_7} = \frac{1}{r_5} + \frac{1}{r_6} \,. \tag{1.24}$$

1.4 Irreducibility and inequivalence

We have now shown how to define a linear representation of the group S_n from a Young diagram, with $n = 13$ in our example. It remains to prove that this representation is irreducible. We also want to prove that the irreducible representations of S_n constructed from different Young diagrams are all inequivalent.

The proof is by induction on n, starting from the trivial case $n = 1$. We assume, as our induction hypothesis, that the different Young diagrams for $n-1$ give inequivalent irreducible representations of S_{n-1}, and we use this assumption to prove that the same result must hold for S_n.

We consider each Young diagram of S_n separately. A central part of the proof is the observation of how this representation of S_n splits into representations of the subgroup S_{n-1}, which are irreducible and inequivalent by the induction hypothesis. Obviously, S_{n-1} regarded as a subgroup of S_n is the fixed point group of the number n.

Let us go back to our example with $n = 13$, with the same Young diagram as before. In order to prove that the corresponding representation of S_{13} is irreducible we may use the converse of Schur's lemma, which holds for finite groups, that a complex representation is irreducible if the only linear transformations commuting with every group element are the multiples of the identity.

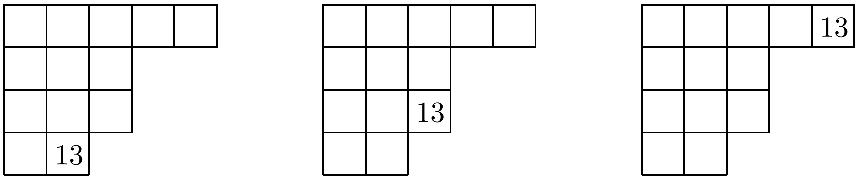


Figure 1.7: The possible positions of the largest number ($n = 13$) in a standard tableau.

A standard Young tableau of this Young diagram may have 13 in three different positions, as shown in Figure 1.7. Removing 13 from a standard tableau of $n = 13$ leaves a standard tableau of $n = 12$. Thus, according to the induction hypothesis, each one of our basis vectors for the representation of S_{13} belongs to an irreducible representation of S_{12}. In particular, in our example, the representation of S_{13} constructed from the given Young diagram splits into a direct sum of three irreducible representations of S_{12}, which are inequivalent by the induction hypothesis.

Our standard ordering of the basis vectors is such that the matrix $\mathbf{D}(p)$ representing the permutation $p \in S_{13}$ is block diagonal whenever $p \in S_{12}$. That is, for $p \in S_{12}$ we have

$$\mathbf{D}(p) = \begin{pmatrix} \mathbf{d}^{(\alpha)}(p) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{d}^{(\beta)}(p) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{d}^{(\gamma)}(p) \end{pmatrix} . \tag{1.25}$$

where $\mathbf{d}^{(\alpha)}, \mathbf{d}^{(\beta)}, \mathbf{d}^{(\gamma)}$ are inequivalent irreducible matrix representations of S_{12}.

Assume now that the matrix $\mathbf{C}$ commutes with every representation matrix $\mathbf{D}(p)$ with $p \in S_{13}$. We want to prove that $\mathbf{C}$ must be proportional to the identity matrix, $\mathbf{C} = c\mathbf{I}$ for some complex number c. We first use the above block diagonal form of $\mathbf{D}(p)$ for $p \in S_{12}$, together with Schur's lemma, to conclude that when $\mathbf{C}$ commutes with every $\mathbf{D}(p)$ with $p \in S_{12}$, it must be block diagonal, with multiples of identity matrices on the diagonal,

$$\mathbf{C} = \begin{pmatrix} c^{(\alpha)}\mathbf{I}^{(\alpha)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & c^{(\beta)}\mathbf{I}^{(\beta)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c^{(\gamma)}\mathbf{I}^{(\gamma)} \end{pmatrix} . \tag{1.26}$$

$$t_u = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 12 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 & 13 \\ \cline{1-3} 10 & 11 \\ \cline{1-2} \end{array} \qquad\qquad t_v = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 13 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 & 12 \\ \cline{1-3} 10 & 11 \\ \cline{1-2} \end{array}$$

Figure 1.8: Two standard tableaux in standard order.

It remains only to prove that the three complex numbers $c^{(\alpha)}, c^{(\beta)}, c^{(\gamma)}$ must be equal. This follows because $\mathbf{C}$ must commute with $\mathbf{D}(p)$ where p is the transposition $T_{12} = (12, 13)$. Consider, again as an example, the two standard tableaux shown in Figure 1.8, transforming as follows,

$$T_{12}\, t_u = -rt_u + qt_v \ , \qquad T_{12}\, t_v = qt_u + rt_v \ , \tag{1.27}$$

with $r = 1/4$ and $q = \sqrt{1 - r^2}$. They belong to, respectively, the irreducible representations β and γ of S_{12}, so that the linear transformation C corresponding to the matrix $\mathbf{C}$ gives

$$Ct_u = c^{(\beta)} t_u \ , \qquad Ct_v = c^{(\gamma)} t_v \ . \tag{1.28}$$

In order to have

$$CT_{12}\, t_u = -rc^{(\beta)} t_u + qc^{(\gamma)} t_v = T_{12}\, Ct_u = -rc^{(\beta)} t_u + qc^{(\beta)} t_v \tag{1.29}$$

we must have $c^{(\beta)} = c^{(\gamma)}$. We prove in a similar way that $c^{(\alpha)} = c^{(\beta)}$, and the proof of irreducibility is complete.

The inequivalence of representations of S_n constructed from different Young diagrams is very easy to prove. We know that they are inequivalent representations of S_n, simply because they are inequivalent representations of the subgroup S_{n-1}, in fact, they split in different ways into irreducible representations of S_{n-1}.

1.5 Character values of S_n are always integers

Two of the irreducible characters of S_n are the two one dimensional representations, the symmetric representation $p \mapsto 1$ and the antisymmetric representation $p \mapsto \operatorname{sgn}(p)$.

If $p \mapsto \rho(p)$ is an irreducible representation of S_n, with character $\chi(p) = \operatorname{Tr} \rho(p)$, then $p \mapsto \widetilde{\rho}(p) = \operatorname{sgn}(p)\rho(p)$ is also an irreducible representation, with character $\widetilde{\chi}(p) = \operatorname{sgn}(p)\chi(p)$. The two representations ρ and $\widetilde{\rho}$ are inequivalent, unless $\chi(p) = 0$ for every odd permutation p. We call $\widetilde{\rho}$ the conjugate of ρ, and it is a natural guess that the two Young diagrams corresponding to conjugate irreducible representations are conjugate diagrams, related by reflection about the diagonal. The proof is left as an exercise.

Thus, the irreducible representations of S_n come in conjugate pairs, except for the self-conjugate representations, with Young diagrams symmetric about the diagonal and with character value zero for all odd permutations.

It simplifies the construction of the character table of S_n to know that the character values have to be integers. We want to prove now this result, and again we proceed by induction. The case of S_1 is trivial. To complete the proof we have to prove that if all the groups S_m with $m \leq n-1$ have integer character values, then the same is true for S_n.

Let χ be the character of an irreducible representation of S_n, different from the two one dimensional representations, which anyway have the integer character values ± 1. Furthermore, let $p \in S_n$ be a permutation which is a product of at least two commuting cycles. An example as good as any is $n = 13$ and

$$p = (1,2,3,4,5,6)(7,8,9)(10,11,12)(13) \; . \tag{1.30}$$

Clearly, p belongs to a subgroup of S_{13} which is the direct product group $S_6 \otimes S_3 \otimes S_3 \otimes S_1$. When we treat a representation of S_{13} as a representation of this subgroup, it is a direct sum of irreducible representations of the subgroup, which in turn are tensor products of the irreducible representations of the groups S_6, S_3, S_3 and S_1. Hence, the character value $\chi(p)$ is a sum of character values for the irreducible representations of the subgroup, but these are all integers, by the induction hypothesis.

One single case is not covered by this argument, and that is when p is an n-cycle. List the conjugation classes of S_n as C_i, with $i = 1,2,\ldots,K$, for example with C_K consisting of all the n-cycles. The irreducible character χ must be orthogonal to the trivial character 1, that is, we must have

$$\sum_{i=1}^{K} N_i \chi_i = 0 \; . \tag{1.31}$$

Here N_i is the number of permutations in the conjugation class C_i, and $\chi(p) = \chi_i$ for $p \in C_i$. In particular, $N_K = (n-1)!$. Since every character value χ_i for $i < K$ is an integer, this orthogonality relation implies that χ_K must be a rational number. But when χ_K is rational it must be an integer, because it is an algebraic integer, like any character value of a finite group (see Appendix A for the definition and properties of algebraic integers). This completes the proof.