Gravitation

Notes for MNFFY250

Jan Myrheim
Department of Physics, NTNU

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Preface: How to read

These notes supplement the text book ("Universe", by R.A. Freedman and W.J. Kaufmann III) in the basic astrophysics course MNFFY250. The book is written mainly for students with little or no background from mathematics and physics, and the notes are intended to cover some additional physical theory, including the mathematics which is part of the theory. The topics covered are basic elements of theoretical astrophysics.

This theory may be familiar to some students taking the course, but unfamiliar to others, especially to first year students, who may find it difficult. I do not want to discourage them from taking the course, but would suggest that they read these notes not for the purpose of understanding and learning everything, but rather with slightly different purposes in mind.

One purpose should be to have a look at the mathematics, if it is unfamiliar, so that it will be less unfamiliar the next time. For example, read the Appendix if the vector notation is a new experience.

Another purpose should be to get some familiarity with the central concepts and results, with some understanding of the logical connections. Here is a partial list of central concepts and results: Force, momentum, kinetic and potential energy, angular momentum; conservation laws; elliptic orbits and semi-major axis; the virial theorem.

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Jan Myrheim

Contents

Chapter 1

Laws of motion

1.1 Newton's laws

Newton's first law

Newton's laws of motion hold in special refence frames called *inertial frames*. By definition, our meter sticks and clocks are located in an inertial frame if we observe that Newton's first law holds:

> *A body remains at rest, or moves in a straight line at constant speed, unless acted upon by a net outside force.*

The first law may be regarded as a special case of the second law: if there is no net force, there is also no acceleration.

In a terrestrial laboratory at rest on the ground we observe that bodies tend to fall to the ground. This does not necessarily mean that the laboratory is not an inertial frame, since we explain the observed downwards acceleration as the effect of an outside force, the gravitational pull of the Earth on the falling body. We may compensate this force by pulling or pushing in the opposite direction, but we can not eliminate it, without removing the whole Earth, or putting our laboratory in free fall. Nevertheless, when due account is taken of the gravitational force, our laboratory on the ground is a reasonably good inertial frame.

It is not a perfect inertial frame, however, because of the rotation of the Earth. If we want to use Newton's law of motion in the laboratory *as if* it were an inertial frame, we may have to introduce two extra forces, the centrifugal force and the Coriolis force, in addition to the gravitational force from the Earth, to account for the observed deviations from straight line motion. The centrifugal and Coriolis forces are not accepted as forces in Newton's sense of the word, because they do not satisfy Newton's third law. They are called *fictitious forces*, and a reference frame in which fictitious forces occur is not an inertial frame.

We obtain a more nearly perfect inertial frame by doing experiments in free fall, for example inside a space station orbiting the Earth, making sure that the space station does not rotate. Even in this laboratory we have to take into account residual gravitational forces, so called *tidal forces*, if we do extremely precise experiments. They are due to the fact that the gravitational field of the Earth is not perfectly homogeneous: there is a tiny variation in the size and direction of the gravitational force, even over the small distance from one point to another inside a space station.

Newton's second law

Newton's second law is the equation of motion for a point-like particle,

$$\boldsymbol{F} = m\boldsymbol{a} \; . \tag{1.1}$$

Here m is the mass (the *inertial mass*) of the particle, and $\boldsymbol{F}$ is the force acting on it. Let $\boldsymbol{r} = \boldsymbol{r}(t)$ denote the time dependent position of the particle, then $\boldsymbol{v} = \dot{\boldsymbol{r}} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ is the velocity, and $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}$ is the acceleration. Each dot over a symbol denotes a differentiation with respect to the time t.

Note that Newton's second law is a vector equation, it consists of three separate equations, one for each of the x, y, z components,

$$F_x = ma_x = m\ddot{x} \; , \qquad F_y = ma_y = m\ddot{y} \; , \qquad F_z = ma_z = m\ddot{z} \; . \tag{1.2}$$

Newton's third law

Newton's third law is the law of action and reaction:

> *Whenever one body exerts a force on a second body, the second body exerts an equal and opposite force on the first body.*

If the force on the second body from the first one is $\boldsymbol{F}_{21}$, and the force on the first body from the second one is $\boldsymbol{F}_{12}$, Newton's third law is the relation

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} \; . \tag{1.3}$$

An important point which is not always clearly stated is that all forces observed in nature are two-body forces. If there are more than two bodies present, then the force between two of them does not depend on the presence of the others, and the force on one body is the sum of the two-body forces from all the others. Hence, Newton's third law implies that the sum of all *internal forces* in a physical system, i.e. all the forces between the particles in the system, is zero.

Fictitious forces

It is sometimes convenient to work in a rotating coordinate system, for example following the rotation of the Earth, even though it is not an inertial system. Then we have to modify Newton's second law by including the centrifugal and Coriolis forces. We call them non-Newtonian, or fictitious, forces, because they have no reaction forces equal in magnitudes and opposite in directions.

Let $\boldsymbol{\omega}$ be a vector along the rotation axis of the Earth, such that $\omega = |\boldsymbol{\omega}|$ is the angular velocity (rotation angle divided by time), and let $\boldsymbol{r}_0$ be a point on the rotation axis, for example the centre of mass of the Earth. Assume that a pointlike particle of mass m is located at the position $\boldsymbol{r}$ and moving with the velocity $\boldsymbol{v}$. Then the centrifugal force is

$$\boldsymbol{F}_{\mathrm{cf}} = -m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_0)) \; , \tag{1.4}$$

and the Coriolis force is

$$\boldsymbol{F}_{\mathrm{C}} = -2m\,\boldsymbol{\omega} \times \boldsymbol{v} \; . \tag{1.5}$$

The Coriolis force is one possible way to explain the observation made by Foucault, that when a pendulum oscillates during several hours, its plane of oscillation rotates. The oscillation plane of a pendulum on the North Pole would be fixed relative to the distant stars, so that relative to the Earth it would rotate by 360 degrees during 24 hours.

Einstein based his general theory of relativity on the postulate that the fictitious forces occurring in non-inertial reference frames are in principle no different from the more "respectable" gravitational force. Thus, the special status of inertial frames is to some degree a matter of convention. In the general theory of relativity physical laws have to be formulated mathematically in such a way that they have the same mathematical form in all reference frames, not only in inertial frames.

Newton's law of universal gravitation

The gravitational force on a pointlike particle of mass m_1 from another pointlike particle of mass m_2 is

$$\boldsymbol{F}_{12} = -\frac{G m_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2)\;. \tag{1.6}$$

Here $G = 6.6726 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ is Newton's gravitational constant.

This law agrees with Newton's third law, since it predicts that the force on the second particle from the first one is

$$\boldsymbol{F}_{21} = -\frac{G m_2 m_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}\,(\boldsymbol{r}_2 - \boldsymbol{r}_1) = \frac{G m_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2) = -\boldsymbol{F}_{12}\;. \tag{1.7}$$

The gravitational force is a *central force*, that is, its direction is precisely towards the attracting particle. The absolute value (the size) of the gravitational force between the two point masses is

$$F = |\boldsymbol{F}_{12}| = |\boldsymbol{F}_{21}| = \frac{G m_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\,|\boldsymbol{r}_1 - \boldsymbol{r}_2| = \frac{G m_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}\;. \tag{1.8}$$

It is proportional to each of the two masses, and it is inversely proportional to the square of the distance between the masses, in other words, it is an "inverse square" law.

According to Newton's law of gravitation the gravitational force acts *instantaneously* over a finite distance. This is a fundamental flaw in the theory, which Newton himself recognized. The problem became even more serious after Einstein developed the special theory of relativity, which postulates that nothing, at least no signal transmitting information, can propagate with a speed larger than the vacuum speed of light. Einstein replaced Newton's theory with a new gravitational theory, the general theory of relativity, in order to repair this basic flaw. Except for mentioning black holes we will say very little about Einstein's gravitational theory here.

1.2 Momentum

Introducing the momentum

$$\boldsymbol{p} = m\boldsymbol{v} \ , \tag{1.9}$$

and using that the mass m is constant, we may write Newton's second law in the following way,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F} \ . \tag{1.10}$$

That is, the force $\boldsymbol{F}$ acting during the infinitesimal time interval $\mathrm{d}t$ changes the momentum by the infinitesimal amount

$$\mathrm{d}\boldsymbol{p} = \boldsymbol{F}\,\mathrm{d}t \ . \tag{1.11}$$

When the time dependent force $\boldsymbol{F} = \boldsymbol{F}(t)$ acts during a finite time interval from t_1 to t_2, the momentum changes from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$, where

$$\boldsymbol{p}_2 - \boldsymbol{p}_1 = \int_{\boldsymbol{p}_1}^{\boldsymbol{p}_2} \mathrm{d}\boldsymbol{p} = \int_{t_1}^{t_2} \boldsymbol{F}\,\mathrm{d}t \ . \tag{1.12}$$

1.3 Kinetic energy

A force acting on a moving particle performs a *work*, and as a result the *energy* of the particle increases by an amount equal to the work. The energy is increased by a positive work of an external force, and reduced by a negative work.

When the force $\boldsymbol{F}$ acts during an infinitesimal displacement $\mathrm{d}\boldsymbol{r}$, it performs an infinitesimal work $\mathrm{d}W$ equal to the scalar product of the force and the displacement,

$$\mathrm{d}W = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z \ . \tag{1.13}$$

Introducing the velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, we may write the displacement as $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$. During a finite time interval from t_1 to t_2, when the particle moves from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$, the work is

$$W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_{t_1}^{t_2} \boldsymbol{F}\cdot\boldsymbol{v}\,\mathrm{d}t \ . \tag{1.14}$$

If $\boldsymbol{F}$ is the total force (the vector sum of all forces) on the particle, then $\boldsymbol{F} = m\boldsymbol{a}$, by Newton's second law, and hence the work is

$$W = \int_{t_1}^{t_2} m\boldsymbol{a}\cdot\boldsymbol{v}\,\mathrm{d}t = \int_{t_1}^{t_2} m\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\boldsymbol{v}\,\mathrm{d}t = \frac{1}{2}\,m\boldsymbol{v}\cdot\boldsymbol{v}\Big|_{t_1}^{t_2} \ . \tag{1.15}$$

We define the *kinetic energy* of the particle as

$$E_K = \frac{1}{2}\,m\boldsymbol{v}\cdot\boldsymbol{v} = \frac{1}{2}\,m\boldsymbol{v}^2 = \frac{1}{2}\,mv^2 = \frac{1}{2}\,m({v_x}^2 + {v_y}^2 + {v_z}^2) \ . \tag{1.16}$$

With this definition, the change in the kinetic energy of the particle equals the work performed by the total force.

In particular, for a free particle, which is not acted upon by any force, the mass and the velocity are both constants of motion. To be more precise, each of the three velocity components v_x, v_y, v_z is a constant of motion. It follows that both the momentum and the kinetic energy of a free particle are constants of motion.

1.4 Relativistic momentum and energy

The above expressions for momentum and energy are valid for a non-relativistic particle. We say that a particle is relativistic when the absolute value of its velocity, $v = |\boldsymbol{v}|$, approaches the speed of light, $c = 299\,792\,458$ m/s, which is the absolute speed limit.

The formulation of Newton's second law in terms of momentum, Equation (1.10), has the advantage that it is valid even for relativistic particles, after we modify the definition of momentum. The definition valid for relativistic as well as for non-relativistic particles, is

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = m\boldsymbol{v}\left(1 + \frac{v^2}{2c^2} + \cdots\right) . \tag{1.17}$$

Expanding to lowest order in the ratio v/c, we get back the non-relativistic definition $\boldsymbol{p} = m\boldsymbol{v}$.

It is not unusual to read the relativistic momentum as $\boldsymbol{p} = m'\boldsymbol{v}$, where the mass m' depends on the velocity,

$$m' = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} , \tag{1.18}$$

and m is called the rest mass (the mass when the particle is at rest). This is of course a possible definition, but it is not recommended (it seems more confusing than useful to distinguish between "rest mass" and "mass in motion").

The relativistic version of Newton's second law is very well tested experimentally. For example, it was used for computing the orbits of electrons in the electric and magnetic fields inside the LEP accelerator ("Large Electron Positron collider", now closed down) at CERN, in Geneva, on the border between France and Switzerland. The electrons in LEP reached a velocity of $0.999\,999\,999\,995\,c$, which means that in one second they would lose 5 mm on a photon.

The above non-relativistic expression for the kinetic energy E_K may be derived in the following way, which leads to the relativistic expression for energy when we introduce the relativistic momentum. By Newton's second law, we have that

$$\boldsymbol{F} \cdot \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \cdot \boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p} \cdot \boldsymbol{v}) - \boldsymbol{p} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} . \tag{1.19}$$

This is true both with the non-relativistic and the relativistic formula for the momentum. With the non-relativistic formula we have that

$$\boldsymbol{p} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{v} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{v} \cdot \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \cdot \boldsymbol{F} = \boldsymbol{F} \cdot \boldsymbol{v} , \tag{1.20}$$

and hence, as before,

$$\boldsymbol{F} \cdot \boldsymbol{v} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p} \cdot \boldsymbol{v}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\boldsymbol{v}^2\right) . \tag{1.21}$$

With the relativistic formula we have that

$$\boldsymbol{p} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}\left(\boldsymbol{v} \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right) = -mc^2\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{1 - \frac{v^2}{c^2}} , \tag{1.22}$$

and hence,

$$\boldsymbol{F} \cdot \boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{p} \cdot \boldsymbol{v} + mc^2 \sqrt{1 - \frac{v^2}{c^2}} \right) = \frac{\mathrm{d}E}{\mathrm{d}t} \,, \tag{1.23}$$

when we introduce the famous relativistic formula for the energy E,

$$E = \boldsymbol{p} \cdot \boldsymbol{v} + mc^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + mc^2 \sqrt{1 - \frac{v^2}{c^2}} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \,. \tag{1.24}$$

Expansion in powers of the ratio v/c gives that

$$E = mc^2 \left(1 + \frac{v^2}{2c^2} + \cdots \right) = mc^2 + \frac{1}{2} mv^2 + \cdots \,. \tag{1.25}$$

The kinetic energy is the energy E minus the rest energy mc^2,

$$E_K = E - mc^2 = \frac{1}{2} mv^2 + \cdots \,. \tag{1.26}$$

Thus we get a formula for the work which is the same in the non-relativistic and the relativistic cases,

$$W = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \boldsymbol{v} \, \mathrm{d}t = E_K(t_2) - E_K(t_1) \,. \tag{1.27}$$

The only difference is that the formula for the kinetic energy E_K is different.

A useful formula relating relativistic energy and momentum is the following,

$$E^2 = m^2 c^4 + \boldsymbol{p}^2 c^2 \,. \tag{1.28}$$

We will have very little use here for the relativistic momentum and energy, but they are presented so that everybody will recognize them when meeting them.

Chapter 2

The one-particle problem

When studying motion in a gravitational field it is natural to start with one single particle. The one-particle problem may be thought of as the limiting case of the two-particle problem when there are two pointlike masses, one small mass m and one large mass M. We write $m \ll M$ to tell that m is much smaller than M. This is a good approximation to the physical problem of one planet, for example the Earth, moving around the Sun.

Later on, we will see that the general two-particle problem may be reduced to this special one-particle problem, even when the two masses are comparable. The deeper reason that the reduction is possible, is that the total momentum is conserved.

We assume that only the small mass m is moving, while the large mass M is lying at rest all the time. This assumption is consistent with the laws of motion, since the forces on the two masses are equal and opposite, and therefore the large mass will have a much smaller acceleration.

It is natural to choose a coordinate system having its origin at the position of the stationary mass. Thus, when $\boldsymbol{r} = \boldsymbol{r}(t)$ is the position of the small mass m at time t, and $r = |\boldsymbol{r}|$, the gravitational force on this particle is

$$\boldsymbol{F} = -\frac{GMm}{r^3}\,\boldsymbol{r}\ . \tag{2.1}$$

In combination with Newton's second law,

$$\boldsymbol{F} = m\boldsymbol{a} = m\,\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\ , \tag{2.2}$$

this gives the following equation of motion, which is a second order ordinary differential equation,

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{GM}{r^3}\,\boldsymbol{r}\ . \tag{2.3}$$

We see that the small mass m cancels out of the equation of motion: the motion of a small mass m in the gravitational field from a very much larger mass M is independent of m. This is a non-trivial result. In fact, the role of the mass m in Equation (2.1) is that the gravitational force is proportional to it. This type of mass may be called *gravitational mass*, or *heavy mass*, it is the mass we feel when we hold a stone in our hand. The mass m in Equation (2.2) plays a very different role, it determines how hard it is to change the state of motion of the particle. This type of mass is called *inertial mass*, it is the mass we feel at the moment when we throw the same stone.

It was one of Newton's great discoveries that these two types of mass are proportional, so that he was able to define them to be the same, by introducing the proportionality constant G in the formula for the gravitational force. He tested this prediction of his theory in the gravitational field of the Earth by comparing the oscillation periods of two pendulums of identical length and shape, made of different substances and having different masses. His experiment was a null experiment, there should be no difference if his hypothesis was correct, and indeed he observed no difference.

The right hand side of Equation (2.3) is what we call the *gravitational field*, or equivalently, the *acceleration of gravity*, due to the point mass M placed at the origin. It is a vector field,

$$\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{r}) = -\frac{GM}{r^3}\,\boldsymbol{r}\ . \tag{2.4}$$

In general, the procedure for measuring a gravitational field is to measure the force on a so called *test particle*, which is supposed to be pointlike, and to have a sufficiently small mass m, so that it has a negligible influence on the motion of the masses giving rise to the gravitational field. It follows from Newton's law of universal gravitation that the gravitational force $\boldsymbol{F}$ on such a test particle is proportional to its mass m, therefore it is natural to define the gravitational field at a given point as the gravitational force divided by the mass,

$$\boldsymbol{g} = \frac{\boldsymbol{F}}{m}\ . \tag{2.5}$$

The acceleration $\boldsymbol{a}$ of the test mass is given by Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$, thus we see that $\boldsymbol{g} = \boldsymbol{a}$, the gravitational field at the given point is simply the acceleration of any test mass placed there.

> *All particles of sufficiently small mass are subject to the same acceleration in a gravitational field.*

2.1 Circular motion

We will now demonstrate that circular motion in any plane through the origin, with any given constant radius r, and a constant angular velocity ω depending on r, is a possible solution of the equation of motion. We choose our coordinate system with the z axis orthogonal to the plane, so that the orbital plane is the (x, y) plane. Thus, the position at time t will be

$$\boldsymbol{r}(t) = x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} = r\cos(\omega t)\,\boldsymbol{i} + r\sin(\omega t)\,\boldsymbol{j}\ , \tag{2.6}$$

if we choose the zero point of time in such a way that $\boldsymbol{r}(t) = r\,\boldsymbol{i}$ at $t = 0$. Differentiating once with respect to time we get the velocity $\boldsymbol{v}$, and differentiating once more we get the acceleration $\boldsymbol{a}$,

$$\begin{aligned} \boldsymbol{v} &= \dot{\boldsymbol{r}} = -\omega r\cos(\omega t)\,\boldsymbol{i} + \omega r\cos(\omega t)\,\boldsymbol{j}\ , \\ \boldsymbol{a} &= \dot{\boldsymbol{v}} = -\omega^2 r\cos(\omega t)\,\boldsymbol{i} - \omega^2 r\sin(\omega t)\,\boldsymbol{j} = -\omega^2\,\boldsymbol{r}\ . \end{aligned} \tag{2.7}$$

Now Newton's second law $\boldsymbol{F} = m\boldsymbol{a}$ takes the form

$$-\frac{GMm}{r^3}\,\boldsymbol{r} = -m\omega^2\,\boldsymbol{r}\ . \tag{2.8}$$

We see that this equation of motion is solved if we take

$$\omega^2 = \frac{GM}{r^3} . \tag{2.9}$$

This relation is one form of Kepler's third law for the special case of circular motion. The period P of the orbit is the time interval for which

$$\omega P = 2\pi . \tag{2.10}$$

Thus we get Kepler's third law in its usual form, as a relation between the period P and the semimajor axis of an ellipse, in this particular case the radius r of a circle,

$$P^2 = \frac{4\pi^2}{\omega^2} = \frac{4\pi^2}{GM} r^3 . \tag{2.11}$$

2.2 Conservation of energy

The gravitational field of the point mass M,

$$\boldsymbol{g} = -\frac{GM}{r^3} \boldsymbol{r} , \tag{2.12}$$

is a vector field which is minus the gradient of a scalar field ϕ, called the *gravitational potential.*

To see this, we compute first the gradient of the distance to the origin, which is the scalar function

$$r = \sqrt{x^2 + y^2 + z^2} . \tag{2.13}$$

We have that

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} 2x = \frac{x}{r} , \tag{2.14}$$

with similar results for $\partial r / \partial y$ and $\partial r \partial z$. Hence,

$$\nabla r = \boldsymbol{i} \frac{\partial r}{\partial x} + \boldsymbol{j} \frac{\partial r}{\partial y} + \boldsymbol{k} \frac{\partial r}{\partial z} = \boldsymbol{i} \frac{x}{r} + \boldsymbol{j} \frac{y}{r} + \boldsymbol{k} \frac{z}{r} = \frac{\boldsymbol{r}}{r} = \boldsymbol{e}_r . \tag{2.15}$$

At any given point, $\boldsymbol{e}_r$ is the unit vector pointing away from the origin. Using the chain rule, we get that

$$\frac{\partial}{\partial x} \left(\frac{1}{r} \right) = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} \left(\frac{1}{r} \right) = -\frac{x}{r} \frac{1}{r^2} = -\frac{x}{r^3} . \tag{2.16}$$

This is the x component of the vector equation

$$\nabla \left(\frac{1}{r} \right) = -\frac{\boldsymbol{r}}{r^3} . \tag{2.17}$$

It follows that

$$\boldsymbol{g} = -\nabla \phi \tag{2.18}$$

when we define the *gravitational potential* from the point mass M at the origin as

$$\phi = -\frac{GM}{r} \; . \tag{2.19}$$

It follows further that the gravitational force on the small mass m is

$$\boldsymbol{F} = m\boldsymbol{g} = -m\nabla\phi = -\nabla V \; , \tag{2.20}$$

when we define the *potential energy* of the mass m in the gravitational field as

$$V = m\phi = -\frac{GMm}{r} \; . \tag{2.21}$$

The motivation for introducing the potential energy V is that the sum of the kinetic and potential energy, the *total mechanical energy*

$$E = E_K + V = \frac{1}{2}m\boldsymbol{v}^2 - \frac{GMm}{r} \; , \tag{2.22}$$

is a constant of motion.

To prove this, we have to prove that the time derivative of E vanishes,

$$\dot{E} = \dot{E}_K + \dot{V} = 0 \; . \tag{2.23}$$

(We write $\dot{E} = \mathrm{d}E/\mathrm{d}t$ for the time derivative of E, and so on.) We calculate

$$\dot{E}_K = \frac{1}{2}m(\dot{\boldsymbol{v}} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \dot{\boldsymbol{v}}) = m\dot{\boldsymbol{v}} \cdot \boldsymbol{v} = m\boldsymbol{a} \cdot \boldsymbol{v} = \boldsymbol{F} \cdot \boldsymbol{v} = -\frac{GMm}{r^3}\boldsymbol{r} \cdot \boldsymbol{v} \; , \tag{2.24}$$

and

$$\dot{V} = \frac{GMm}{r^2}\dot{r} = \frac{GMm}{r^2}\frac{\boldsymbol{r} \cdot \boldsymbol{v}}{r} \; , \tag{2.25}$$

which proves Equation (2.23). In the calculation of $\dot{V}$ we may use the following shortcut to show that $\dot{r} = \boldsymbol{r} \cdot \boldsymbol{v}/r$. We differentiate both sides of the identity $r^2 = \boldsymbol{r} \cdot \boldsymbol{r}$, this gives that

$$2r\dot{r} = \dot{\boldsymbol{r}} \cdot \boldsymbol{r} + \boldsymbol{r} \cdot \dot{\boldsymbol{r}} = 2\boldsymbol{r} \cdot \dot{\boldsymbol{r}} = 2\boldsymbol{r} \cdot \boldsymbol{v} \; . \tag{2.26}$$

We may also calculate $\dot{E}$ in the following way, which gives the same result, but shows even more clearly what is really going on. Here we use Newton's second law in the form $m\boldsymbol{a} = -\nabla V$, and we use the chain rule to calculate $\dot{V}$,

$$\dot{E} = \dot{E}_K + \dot{V} = m\dot{\boldsymbol{v}} \cdot \boldsymbol{v} + (\nabla V) \cdot \dot{\boldsymbol{r}} = (m\boldsymbol{a} + \nabla V) \cdot \boldsymbol{v} = 0 \; . \tag{2.27}$$

We see that we have proved a very general result: the total (mechanical) energy $E = E_K + V$ is a conserved quantity (a constant of motion) whenever the force field $\boldsymbol{F}$ is minus the gradient of a potential energy function V, that is, when Newton's second law holds in the form $m\boldsymbol{a} = -\nabla V$, and the potential energy function has no explicit time dependence, that is, we have that $V = V(\boldsymbol{r})$ and not $V = V(\boldsymbol{r}, t)$. We have proved this result here for one particle, but it is easily generalized to any number of particles.

2.3 Conservation of angular momentum

We define the *angular momentum* of the particle as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (m\boldsymbol{v}) \; . \tag{2.28}$$

It is easy to show that the angular momentum of a point particle is a constant of motion when the particle is moving in a central force field, that is, when the force $\boldsymbol{F}$ always points along $\boldsymbol{r}$. We just compute the time derivative of $\boldsymbol{L}$ and find that it vanishes,

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{r} \times \boldsymbol{F} = 0 + 0 = 0 \; . \tag{2.29}$$

The conservation law for the vector $\boldsymbol{L}$ is actually a combination of three conservation laws, one for each of the three components

$$\begin{aligned} L_x &= yp_z - zp_y = m(yv_z - zv_y) \; , \\ L_y &= zp_x - xp_z = m(zv_x - xv_z) \; , \\ L_z &= xp_y - yp_x = m(xv_y - yv_x) \; . \end{aligned} \tag{2.30}$$

Note that

$$\boldsymbol{L} = m \, \frac{\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \; . \tag{2.31}$$

The length $|\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}|$ of the infinitesimal vector $\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}$ can be interpreted geometrically as an area, it is twice the area swept out by the radius vector $\boldsymbol{r}$ during the infinitesimal time interval $\mathrm{d}t$. This shows that Kepler's second law is a consequence of the conservation law for angular momentum.

2.4 The general Kepler orbit

In order to find the general solution of the equation of motion for our one-particle problem, we use the conservation laws for energy and angular momentum. We assume that $\boldsymbol{L} \neq 0$, because if $\boldsymbol{L} = 0$, it means that the velocity $\boldsymbol{v}$ is along the radius vector $\boldsymbol{r}$, so that the particle is going to hit the point mass at the origin. A planet with zero angular momentum would crash into the Sun.

Reduction to two dimensions

Since $\boldsymbol{r} \cdot \boldsymbol{L} = \boldsymbol{r} \cdot (\boldsymbol{r} \times \boldsymbol{p}) = 0$, and since $\boldsymbol{L}$ is a constant vector, we conclude that the particle moves all the time in the plane which goes through the origin and is orthogonal to $\boldsymbol{L}$. Therefore we choose an (x, y, z) coordinate system with its z axis along $\boldsymbol{L}$. In this coordinate system, the particle moves in the (x, y) plane, it has always $z = 0$, and its angular momentum components are $L_x = L_y = 0$, $L_z = L = |\boldsymbol{L}| > 0$.

The position of the particle at time t is

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} = r\cos\varphi\,\boldsymbol{i} + r\sin\varphi\,\boldsymbol{j} = r\,\boldsymbol{e}_r \; . \tag{2.32}$$

The Cartesian coordinates (x, y) and the polar coordinates (r, φ) are all time dependent: $x = x(t)$, $y = y(t)$, $r = r(t)$, and $\varphi = \varphi(t)$. We introduce the unit vector in the direction along $\boldsymbol{r}$,

$$\boldsymbol{e}_r = \cos\varphi\, \boldsymbol{i} + \sin\varphi\, \boldsymbol{j} \ , \tag{2.33}$$

which is also time dependent, since $\varphi = \varphi(t)$. The unit vector orthogonal to $\boldsymbol{e}_r$, in the same plane, is

$$\boldsymbol{e}_\varphi = -\sin\varphi\, \boldsymbol{i} + \cos\varphi\, \boldsymbol{j} \ . \tag{2.34}$$

In order to simplify the notation we denote time derivatives by dots,

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} \ , \qquad \dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} \ . \tag{2.35}$$

We have that

$$\dot{\boldsymbol{e}}_r = -(\sin\varphi)\dot{\varphi}\, \boldsymbol{i} + (\cos\varphi)\dot{\varphi}\, \boldsymbol{j} = \dot{\varphi}\, \boldsymbol{e}_\varphi \ . \tag{2.36}$$

With this notation the velocity is

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\, \boldsymbol{e}_r + r\, \dot{\boldsymbol{e}}_r = \dot{r}\, \boldsymbol{e}_r + r\dot{\varphi}\, \boldsymbol{e}_\varphi \ . \tag{2.37}$$

Since $\boldsymbol{e}_r$ and $\boldsymbol{e}_\varphi$ are orthogonal unit vectors, we have that

$$\boldsymbol{v} \cdot \boldsymbol{v} = \dot{r}^2 + r^2\dot{\varphi}^2 \ , \tag{2.38}$$

and the energy is

$$E = \frac{1}{2}\, m(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{GMm}{r} \ . \tag{2.39}$$

Since

$$\boldsymbol{e}_r \times \boldsymbol{e}_r = 0 \ , \qquad \boldsymbol{e}_r \times \boldsymbol{e}_\varphi = \boldsymbol{k} \ , \tag{2.40}$$

the angular momentum is

$$\boldsymbol{L} = m(\boldsymbol{r} \times \boldsymbol{v}) = mr^2\dot{\varphi}\, \boldsymbol{k} \ . \tag{2.41}$$

And since we have chosen our coordinate system in such a way that $\boldsymbol{L} = L_z\, \boldsymbol{k} = L\, \boldsymbol{k}$, where $L = |\boldsymbol{L}|$, we have that

$$L = mr^2\dot{\varphi} \ . \tag{2.42}$$

Reduction to one dimension

Instead of using Newton's second law directly, we use the conservation laws for the energy E and the angular momentum L. A conservation law is a partial solution of the equation of motion (sometimes even a complete solution), we say that it is a *first integral* of the equation of motion.

First we use the conservation law for angular momentum to express the time derivative of the polar angle φ in the following way,

$$\dot{\varphi} = \frac{L}{mr^2} \ . \tag{2.43}$$

Next we eliminate $\dot{\varphi}$ in the conservation law for energy, to obtain the equation

$$E = \frac{1}{2}\, m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \ . \tag{2.44}$$

Remember that E, L, G, M, and m are constants.

This is precisely the equation of motion for a particle in one dimension, having an "effective potential energy"

$$V_{1d} = V_{1d}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \ . \tag{2.45}$$

In addition to the gravitational potential energy $V = -GMm/r$, inversely proportional to r, there appears the so called "centrifugal potential energy" $L^2/(2mr^2)$, inversely proportional to r^2.

We see that $V_{1d} \to 0$ as $r \to \infty$, that $V_{1d} < 0$ for $r > L^2/(2GMm^2)$, and that $V_{1d} \to +\infty$ as $r \to 0$. For a given angular momentum L the one dimensional potential energy V_{1d} has a minimum value at a distance $r = r_0$ which is the solution of the equation

$$\frac{\mathrm{d}V_{1d}}{\mathrm{d}r} = -\frac{L^2}{mr^3} + \frac{GMm}{r^2} = 0 \ . \tag{2.46}$$

Thus,

$$r_0 = \frac{L^2}{GMm^2} \ . \tag{2.47}$$

At the distance $r = r_0$ the gravitational potential energy is

$$V_0 = V(r_0) = -\frac{GMm}{r_0} = -\frac{G^2M^2m^3}{L^2} \ . \tag{2.48}$$

Adding the centrifugal energy we get the minimum value of V_{1d},

$$V_{1d0} = V_{1d}(r_0) = \frac{L^2}{2mr_0^2} - \frac{GMm}{r_0} = \frac{L^2}{2mr_0}\,\frac{GMm^2}{L^2} - \frac{GMm}{r_0} = -\frac{GMm}{2r_0} = \frac{V_0}{2} \ . \tag{2.49}$$

The special case of circular motion

Since the one dimensional kinetic energy $(1/2)m\dot{r}^2$ is never negative, $V_{1d0} = V_0/2$ is also the lower limit to the total energy E, given the angular momentum L. For any given value of the energy E (with $E \geq V_0/2$) the distance to the origin, r, has a positive lower limit. If $E < 0$ there is also an upper limit for r, which means that the particle is bound and can never escape to infinity.

Clearly $r = r_0 =$ constant is a solution of the equation of motion such that the particle moves in a circle. In this circular orbit the total energy is

$$E_0 = V_{1d0} = \frac{V_0}{2} \ . \tag{2.50}$$

This is an interesting result. In a circular orbit the total energy E is constant, and is exactly half of the potential energy V, which is also constant,

$$E = E_K + V = \frac{V}{2} \, . \tag{2.51}$$

Another formulation of the same result is that the kinetic energy E_K is half the absolute value of the potential energy,

$$E_K = E - V = -\frac{V}{2} = \frac{|V|}{2} \, . \tag{2.52}$$

It is actually a special case of a much more general result, called the virial theorem, which we will prove later.

The general motion

Equation (2.44) is a first order ordinary differential equation for r as a function of t, it is separable and can therefore be solved explicitly.

However, it turns out that we get a simpler equation by means of two special tricks. The first trick is to solve for r as a function of φ instead of t. By the chain rule for differentiation we get that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\dot{\varphi} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\,\frac{L}{mr^2} \, . \tag{2.53}$$

Hence the energy is

$$E = \frac{L^2}{2m}\left(\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2}\right) - \frac{GMm}{r} \, . \tag{2.54}$$

The second trick is to introduce a new variable

$$u = \frac{1}{r} \, . \tag{2.55}$$

Since $r = 1/u$, and

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi} = -r^2\,\frac{\mathrm{d}u}{\mathrm{d}\varphi} \, , \tag{2.56}$$

the energy is

$$E = \frac{L^2}{2m}\left(\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2\right) - GMmu \, . \tag{2.57}$$

We differentiate this equation with respect to φ, and get that

$$0 = \frac{L^2}{m}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u\,\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right) - GMm\,\frac{\mathrm{d}u}{\mathrm{d}\varphi} \, . \tag{2.58}$$

To satisfy this equation we must have either $\mathrm{d}u/\mathrm{d}\varphi = 0$, which is once more the case of circular motion, or else

$$0 = \frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u - \frac{GMm^2}{L^2} \, . \tag{2.59}$$

The most general solution of the last equation is

$$u = u_0 - A\cos(\varphi - \varphi_0) \; , \tag{2.60}$$

where A and φ_0 are arbitrary integration constants, and u_0 is a constant which is *not* arbitrary,

$$u_0 = \frac{1}{r_0} = \frac{GMm^2}{L^2} \; . \tag{2.61}$$

We may always choose $A \geq 0$, since we get the same solution by swithching the sign of A while adding π to φ_0. We should check explicitly that Equation (2.60) is a solution of Equation (2.57), and we find that it is, when

$$E = \frac{L^2}{2m}\left(A^2 + u_0^2\right) - GMmu_0 = \frac{L^2}{2m}\left(A^2 - u_0^2\right) \; . \tag{2.62}$$

There are many other ways to write this expression for the energy, for example,

$$E = \frac{L^2}{2m}\left(A^2 - \frac{1}{r_0^2}\right) = \frac{L^2A^2}{2m} - \frac{GMm}{2r_0} \; . \tag{2.63}$$

Equation (2.60) is the same as the equation

$$r = \frac{1}{u_0 - A\cos(\varphi - \varphi_0)} \; . \tag{2.64}$$

We see that if $0 \leq A < u_0$, then r oscillates between a minimum value, for $\cos\varphi = -1$, and a maximum value, for $\cos\varphi = 1$,

$$r_{\text{min}} = \frac{1}{u_0 + A} \; , \qquad r_{\text{max}} = \frac{1}{u_0 - A} \; . \tag{2.65}$$

This means that the particle is bound, and by Equation (2.62) the energy of a bound orbit is negative, $E < 0$.

In the case of a planet orbiting the Sun, the point of minimum distance is called *perihelion*, and the point of maximum distance is called *aphelion*. Assuming that the orbit is an ellipse, the semi-major axis of the ellipse must be

$$a = \frac{r_{\text{min}} + r_{\text{max}}}{2} = \frac{u_0}{u_0^2 - A^2} \; . \tag{2.66}$$

The special case $A = 0$ is the same circular solution that we found earlier,

$$r = \frac{1}{u_0} = r_0 = \frac{L^2}{GMm^2} \; . \tag{2.67}$$

The integration constant φ_0 is an angle which determines the orientation of the orbit in the (x, y) plane. From now on we will assume that $\varphi_0 = 0$, this means simply that we choose suitable coordinate axes in the plane. Since $x = r\cos\varphi$, our solution (2.60) with $\varphi_0 = 0$ may be written as

$$u_0 r - Ax = 1 \; . \tag{2.68}$$

We will see now that this is the equation of an ellipse when $0 < A < u_0$.

Let us derive the equation for an ellipse with one focus at the origin and the other focus at $y = 0$, $x = 2ea$, where a is the semi-major axis and e the *eccentricity* of the ellipse. The definition of an ellipse is that the sum of the distances to the two foci is constant, and the constant is equal to $2a$, twice the semi-major axis. That is, a point (x, y) on the ellipse satisfies the equation

$$r + \sqrt{(x - 2ea)^2 + y^2} = 2a \;, \tag{2.69}$$

with $r = \sqrt{x^2 + y^2}$, or

$$\sqrt{x^2 - 4eax + 4e^2a^2 + y^2} = 2a - r \;. \tag{2.70}$$

Squaring this equation, then subtracting the identity $x^2 + y^2 = r^2$ and dividing by $4a$, gives that

$$r - ex = (1 - e^2)a \;. \tag{2.71}$$

We see that the orbit we found, Equation (2.68), with $0 \leq A < u_0$, is an ellipse with eccentricity

$$e = \frac{A}{u_0} = \frac{L^2 A}{GMm^2} \tag{2.72}$$

and semi-major axis

$$a = \frac{1}{u_0(1 - e^2)} = \frac{u_0}{u_0^2 - A^2} \;. \tag{2.73}$$

The energy and period of an elliptical orbit

The energy may now be written as a function of a alone,

$$E = \frac{L^2}{2m}\left(A^2 - u_0^2\right) = -\frac{L^2}{2m}\frac{u_0}{a} = -\frac{L^2}{2m}\frac{GMm^2}{L^2 a} = -\frac{GMm}{2a} = \frac{V(a)}{2} \;. \tag{2.74}$$

Here $V(a) = -GMm/a$ is the potential energy at the distance a.

Incidentally, $V(a)$ is also equal to the time average $\overline{V}$ of the potential energy $V(r) = -GMm/r$ in the elliptical orbit. That $E = V(a)/2$, in other words, that the energy in a bound orbit is half of the time average of the potential energy, is again an example of the virial theorem.

The period of the elliptical orbit is

$$P = \int_0^P \mathrm{d}t = \int_0^P \frac{mr^2}{L}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\,\mathrm{d}t = \frac{m}{L}\int_0^{2\pi} r^2\,\mathrm{d}\varphi = \frac{2m\mathcal{A}}{L} \;, \tag{2.75}$$

where $\mathcal{A}$ is the area of the ellipse. The integral equals $2\mathcal{A}$ because the infinitesimal quantity $r^2\,\mathrm{d}\varphi$ is twice the area of an infinitesimal triangle, and all the infinitesimal triangles together make up the ellipse.

An ellipse with semi-major axis a and semi-minor axis $b = a\sqrt{1-e^2}$ may be regarded as a circle of radius a which has been squeezed in one direction by a factor b/a. Therefore the area of the ellipse is

$$\mathcal{A} = \pi ab = \pi a^2 \sqrt{1-e^2}\,. \tag{2.76}$$

It follows that

$$P^2 = \frac{4\pi^2 m^2 a^4 (1-e^2)}{L^2} = \frac{4\pi^2 m^2 a^3}{L^2 u_0} = \frac{4\pi^2 m^2 a^3}{GMm^2} = \frac{4\pi^2 a^3}{GM}\,, \tag{2.77}$$

and this is Kepler's third law.

The average potential energy

The time average of the potential energy of the particle in an elliptical orbit is

$$\begin{aligned}\overline{V} &= \frac{1}{P}\int_0^P \left(-\frac{GMm}{r}\right)\,\mathrm{d}t = -\frac{1}{P}\int_0^P \frac{GMm}{r}\,\frac{mr^2}{L}\,\frac{\mathrm{d}\varphi}{\mathrm{d}t}\,\mathrm{d}t = -\frac{1}{P}\int_0^{2\pi}\frac{GMm}{r}\,\frac{mr^2}{L}\,\mathrm{d}\varphi \\ &= -\frac{GMm^2}{PL}\int_0^{2\pi} r\,\mathrm{d}\varphi = -\frac{2GMm^2}{PL}\int_0^{\pi} r\,\mathrm{d}\varphi = -\frac{2GMm^2}{PL}\int_0^{\pi}\frac{\mathrm{d}\varphi}{u_0 - A\cos\varphi}\,. \end{aligned} \tag{2.78}$$

The standard trick for solving integrals like this is to introduce a new variable $w = \tan(\varphi/2)$. This gives that

$$\mathrm{d}w = \frac{1}{\cos^2\frac{\varphi}{2}}\,\frac{\mathrm{d}\varphi}{2} = (1+w^2)\,\frac{\mathrm{d}\varphi}{2}\,, \tag{2.79}$$

and

$$\cos\varphi = 2\cos^2\frac{\varphi}{2} - 1 = \frac{2}{1+w^2} - 1 = \frac{1-w^2}{1+w^2}\,, \tag{2.80}$$

Hence,

$$\begin{aligned}\int_0^{\pi}\frac{\mathrm{d}\varphi}{u_0 - A\cos\varphi} &= \int_0^{\infty}\frac{1}{\left(u_0 - A\,\frac{1-w^2}{1+w^2}\right)}\,\frac{2\,\mathrm{d}w}{1+w^2} = 2\int_0^{\infty}\frac{\mathrm{d}w}{u_0 - A + (u_0 + A)w^2} \\ &= \frac{2}{u_0 - A}\sqrt{\frac{u_0 - A}{u_0 + A}}\int_0^{\infty}\frac{\mathrm{d}x}{1+x^2} = \frac{\pi}{\sqrt{u_0^2 - A^2}}\,, \end{aligned} \tag{2.81}$$

where we have introduced the second new variable

$$x = w\sqrt{\frac{u_0 + A}{u_0 - A}}\,. \tag{2.82}$$

Thus,

$$\overline{V} = -\frac{2\pi GMm^2}{PL\sqrt{u_0^2 - A^2}} = -\frac{2\pi GMm^2\sqrt{a}}{PL\sqrt{u_0}} = -\frac{2\pi GMm^2\sqrt{a}}{2\pi m\sqrt{a^3}} = -\frac{GMm}{a}\,. \tag{2.83}$$

2.5 The virial theorem for one particle

The virial theorem for one small pointlike mass m in the gravitational field of a large pointlike mass M at rest at the origin is proved by computing the *virial*, which is the quantity

$$U = \boldsymbol{r} \cdot \boldsymbol{p} = \boldsymbol{r} \cdot (m\boldsymbol{v}) \ . \tag{2.84}$$

First we compute its time derivative, which is

$$\dot{U} = \dot{\boldsymbol{r}} \cdot \boldsymbol{p} + \boldsymbol{r} \cdot \dot{\boldsymbol{p}} = \boldsymbol{v} \cdot \boldsymbol{p} + \boldsymbol{r} \cdot \boldsymbol{F} = \boldsymbol{v} \cdot \boldsymbol{p} - \boldsymbol{r} \cdot (\nabla V) = 2E_K + V \ . \tag{2.85}$$

Here $E_K = (1/2)m\boldsymbol{v}^2 = (1/2)\boldsymbol{p} \cdot \boldsymbol{v}$ is the kinetic energy, and $V = -GMm/r$ is the potential energy. In fact, we have that

$$\boldsymbol{r} \cdot (\nabla V) = \boldsymbol{r} \cdot \left(\frac{GMm}{r^3} \boldsymbol{r} \right) = \frac{GMm}{r^3} \boldsymbol{r} \cdot \boldsymbol{r} = \frac{GMm}{r} = -V \ . \tag{2.86}$$

The relation $\boldsymbol{r} \cdot (\nabla V) = -V$ may be derived not only by direct calculation, as we just did, but also by the following very general argument. We observe that the gravitational potential energy

$$V(\boldsymbol{r}) = V(x, y, z) = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{x^2 + y^2 + z^2}} \tag{2.87}$$

is a *homogeneous function of degree* -1 of its arguments x, y, z. That is, if we scale all three arguments by a common factor $\lambda > 0$, we have that

$$V(\lambda \boldsymbol{r}) = V(\lambda x, \lambda y, \lambda z) = -\frac{GMm}{\lambda r} = \lambda^{-1} \, V(\boldsymbol{r}) \ . \tag{2.88}$$

Taking $\lambda = 1 + \epsilon$, where ϵ is infinitesimal, we get first that, to first order in ϵ,

$$\begin{aligned} V(\lambda \boldsymbol{r}) &= V(x + \epsilon x, y + \epsilon y, z + \epsilon z) \\ &= V(x, y, z) + \epsilon x \, \frac{\partial V(x, y, z)}{\partial x} + \epsilon y \, \frac{\partial V(x, y, z)}{\partial y} + \epsilon z \, \frac{\partial V(x, y, z)}{\partial z} && (2.89) \\ &= V(\boldsymbol{r}) + \epsilon \, \boldsymbol{r} \cdot \nabla V(\boldsymbol{r}) \ . && (2.90) \end{aligned}$$

And second, because V is a homogeneous function of degree -1,

$$V(\lambda \boldsymbol{r}) = (1 + \epsilon)^{-1} \, V(\boldsymbol{r}) = (1 - \epsilon) \, V(\boldsymbol{r}) \ . \tag{2.91}$$

Comparing these two expressions, we conclude that $\boldsymbol{r} \cdot \nabla V = -V$.

The second step in deriving the virial theorem is to take the time average of the equation $\dot{U} = 2E_K + V$, by integrating over a time interval from t_1 to t_2 and dividing by $t_2 - t_1$. Denoting the time average by brackets $\langle \, \rangle$ we get that

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} = 2\langle E_K \rangle + \langle V \rangle \ . \tag{2.92}$$

The left hand side of this equation vanishes in the case where we have a periodic orbit, and we integrate over one or more complete periods. More generally, it vanishes in the limit of an infinitely long time interval, if the system is bound, so that neither the position vector $\boldsymbol{r}$ nor the momentum vector $\boldsymbol{p}$ go to infinity.

The virial theorem, which we have now proved for one particle, is the statement that in a system which is gravitationally bound, the time average (over a long time interval) of twice the kinetic energy plus the potential energy is zero,

$$2\langle E_K \rangle + \langle V \rangle = 0 \; . \tag{2.93}$$

Since the total energy $E = E_K + V$ is constant, it follows that

$$E = \langle E_K \rangle + \langle V \rangle = -\langle E_K \rangle = \frac{\langle V \rangle}{2} \; . \tag{2.94}$$

We will see that the virial theorem holds for a system consisting of any number of particles, when the whole system is gravitationally bound. It has many applications in astrophysics, for example in understanding the stability of a star, or in measuring the total mass of a star cluster or a cluster of galaxies.

Chapter 3

The two-particle problem

Now that we have solved the gravitational one-particle problem which is the special two-particle problem with very unequal masses, it turns out to be easy to solve the general two-particle problem, with masses that may be comparable, by reducing it to the one-particle case. Thus, we assume now that there are two pointlike masses m_1 and m_2, which we allow to be comparable.

3.1 The relative motion

The time dependent positions of the two particles are $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. It is convenient to introduce the relative position, which we define as

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \; . \tag{3.1}$$

The gravitational force on particle 1 from particle 2 is

$$\boldsymbol{F}_{12} = -\frac{Gm_1 m_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2) = -\frac{Gm_1 m_2}{r^3}\,\boldsymbol{r} \; , \tag{3.2}$$

and the force on particle 2 from particle 1 is

$$\boldsymbol{F}_{21} = -\frac{Gm_2 m_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3}\,(\boldsymbol{r}_2 - \boldsymbol{r}_1) = \frac{Gm_2 m_1}{r^3}\,\boldsymbol{r} = -\boldsymbol{F}_{12} \; . \tag{3.3}$$

We asume that there are no other forces acting on the particles. Then Newton's second law applied to each of them gives that

$$\boldsymbol{F}_{12} = m_1 \boldsymbol{a}_1 = m_1\,\frac{\mathrm{d}^2\boldsymbol{r}_1}{\mathrm{d}t^2} \; , \qquad \boldsymbol{F}_{21} = m_2 \boldsymbol{a}_2 = m_2\,\frac{\mathrm{d}^2\boldsymbol{r}_2}{\mathrm{d}t^2} \; . \tag{3.4}$$

These equations of motion are second order ordinary differential equations, one vector equation for each of the particles,

$$\frac{\mathrm{d}^2\boldsymbol{r}_1}{\mathrm{d}t^2} = -\frac{Gm_2}{r^3}\,\boldsymbol{r} \; , \qquad \frac{\mathrm{d}^2\boldsymbol{r}_2}{\mathrm{d}t^2} = \frac{Gm_1}{r^3}\,\boldsymbol{r} \; . \tag{3.5}$$

By subtracting them we get immediately an equation of motion for the relative position,

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\boldsymbol{r}_1}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\boldsymbol{r}_2}{\mathrm{d}t^2} = -\frac{G(m_1+m_2)}{r^3}\,\boldsymbol{r} = -\frac{GM}{r^3}\,\boldsymbol{r} \; , \tag{3.6}$$

where we have introduced the total mass

$$M = m_1 + m_2 \ . \tag{3.7}$$

We have chosen our notation in such a clever way that the equation we obtain is formally exactly the same as the one-particle equation we solved above. Thus, we know already how to solve it.

3.2 The centre of mass and conservation of momentum

By definition, the centre of mass of the two particles has the position

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{M} \ . \tag{3.8}$$

It moves with the velocity

$$\boldsymbol{V} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \frac{m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2}{M} \ . \tag{3.9}$$

The total momentum, which we define as

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2 = M\boldsymbol{V} \ , \tag{3.10}$$

is conserved, as usual, because of Newton's second and third laws,

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{p}_1}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{p}_2}{\mathrm{d}t} = \boldsymbol{F}_{12} + \boldsymbol{F}_{21} = 0 \ . \tag{3.11}$$

The conservation of total momentum means that the centre of mass moves with a constant velocity.

3.3 The reduced mass

Knowing the centre of mass position $\boldsymbol{R}$ and the relative position $\boldsymbol{r}$ is the same as knowing the two particle positions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. In fact, we find easily that

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{M}\boldsymbol{r} \ , \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{M}\boldsymbol{r} \ . \tag{3.12}$$

The same relations hold for the velocities, as we find by taking the time derivatives,

$$\boldsymbol{v}_1 = \boldsymbol{V} + \frac{m_2}{M}\boldsymbol{v} \ , \qquad \boldsymbol{v}_2 = \boldsymbol{V} - \frac{m_1}{M}\boldsymbol{v} \ . \tag{3.13}$$

We may now compute the total kinetic energy,

$$\begin{aligned} E_K &= \frac{1}{2} m_1 \boldsymbol{v}_1^2 + \frac{1}{2} m_2 \boldsymbol{v}_2^2 = \frac{1}{2}(m_1 + m_2)\boldsymbol{V}^2 + \frac{1}{2}\frac{m_1 m_2^2 + m_1^2 m_2}{M^2}\boldsymbol{v}^2 \\ &= \frac{1}{2} M\boldsymbol{V}^2 + \frac{1}{2} m\boldsymbol{v}^2 \ , \end{aligned} \tag{3.14}$$

where we have introduced the *reduced mass*

$$m = \frac{m_1 m_2^2 + m_1^2 m_2}{M^2} = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1 + m_2} \ . \tag{3.15}$$

The reduced mass is half of the so called harmonic mean of the two masses, since

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2} \,. \tag{3.16}$$

In terms of the total mass M and the reduced mass m the gravitational force between the two particles is

$$\boldsymbol{F} = \boldsymbol{F}_{12} = -\boldsymbol{F}_{21} = -\frac{Gm_1m_2}{r^3}\,\boldsymbol{r} = -\frac{GMm}{r^3}\,\boldsymbol{r} \,. \tag{3.17}$$

In summary, what we have done is to separate the two-particle problem into two *independent* one-particle problems. One for the centre of mass $\boldsymbol{R}$, which moves as a free paricle of mass M. And one for the relative position $\boldsymbol{r}$, which moves as a particle of mass m, subject to the force $\boldsymbol{F}$.

3.4 Conservation of energy

Our separation of the two-particle problem into two independent one-particle problems implies that the total mechanical energy

$$E = E_K + V = \frac{1}{2}\,m_1\boldsymbol{v}_1^2 + \frac{1}{2}\,m_2\boldsymbol{v}_2^2 - \frac{Gm_1m_2}{r} \tag{3.18}$$

is conserved. We have in fact seen that we may rewrite it as

$$E = \frac{1}{2}\,M\boldsymbol{V}^2 + \frac{1}{2}\,m\boldsymbol{v}^2 - \frac{GMm}{r} = E_{\text{cm}} + E_{\text{rel}} \,, \tag{3.19}$$

where E_{cm} is the energy of the centre of mass motion, and E_{rel} is the energy of the relative motion,

$$E_{\text{cm}} = \frac{1}{2}\,M\boldsymbol{V}^2 \,, \qquad E_{\text{rel}} = \frac{1}{2}\,m\boldsymbol{v}^2 - \frac{GMm}{r} \,. \tag{3.20}$$

We know already that these two energies are separately conserved. However, we will now also prove directly that the energy of the two-particle system is conserved.

The two-particle potential energy

$$V = V(\boldsymbol{r}_1, \boldsymbol{r}_2) = V(r) = -\frac{GMm}{r} \tag{3.21}$$

gives the force on each of the two particles. To see how, first recall that

$$r = |\boldsymbol{r}_1 - \boldsymbol{r}_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2} \,. \tag{3.22}$$

We now define the gradient operator (nabla operator) with respect to the coordinates of particle 1 as

$$\nabla_1 = \boldsymbol{i}\,\frac{\partial}{\partial x_1} + \boldsymbol{j}\,\frac{\partial}{\partial y_1} + \boldsymbol{k}\,\frac{\partial}{\partial z_1} \,, \tag{3.23}$$

and similarly for particle 2. We have for example that

$$\begin{aligned}
\frac{\partial r}{\partial x_1} &= \frac{1}{2r}\,2(x_1 - x_2) = \frac{x_1 - x_2}{r}\ , \\
\frac{\partial r}{\partial x_2} &= \frac{1}{2r}\,2(x_2 - x_1) = -\frac{x_1 - x_2}{r}\ ,
\end{aligned} \tag{3.24}$$

and so on, for all the partial derivatives. It follows that

$$\begin{aligned}
\nabla_1\, r &= \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{r} = \frac{\boldsymbol{r}}{r} = \boldsymbol{e}_r\ , \\
\nabla_2\, r &= \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{r} = -\frac{\boldsymbol{r}}{r} = -\boldsymbol{e}_r\ .
\end{aligned} \tag{3.25}$$

In this way we find that the force on each of the two particles may be written as a negative gradient of the same potential energy,

$$\boldsymbol{F}_{12} = -\frac{GM}{r^3}\,\boldsymbol{r} = -\nabla_1 V\ , \qquad \boldsymbol{F}_{21} = \frac{GM}{r^3}\,\boldsymbol{r} = -\nabla_2 V\ . \tag{3.26}$$

The proof of energy conservation is essentially the same as in the one-particle case, again we have to prove that the time derivative of E vanishes. We have that

$$\begin{aligned}
\dot{E} &= \dot{E}_K + \dot{V} = m_1\boldsymbol{a}_1\cdot\boldsymbol{v}_1 + m_2\boldsymbol{a}_2\cdot\boldsymbol{v}_2 + (\nabla_1 V)\cdot\boldsymbol{v}_1 + (\nabla_2 V)\cdot\boldsymbol{v}_2 \\
&= (m_1\boldsymbol{a}_1 + \nabla_1 V)\cdot\boldsymbol{v}_1 + (m_2\boldsymbol{a}_2 + \nabla_2 V)\cdot\boldsymbol{v}_2 = 0 + 0 = 0\ .
\end{aligned} \tag{3.27}$$

Hopefully it should now be reasonably clear how to generalize the law of conservation of mechanical energy to the case of more than two particles.

3.5 Conservation of angular momentum

The total angular momentum is the sum of the angular momenta of the two particles,

$$\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2 = \boldsymbol{r}_1\times\boldsymbol{p}_1 + \boldsymbol{r}_2\times\boldsymbol{p}_2 = \boldsymbol{r}_1\times(m_1\boldsymbol{v}_1) + \boldsymbol{r}_2\times(m_2\boldsymbol{v}_2)\ . \tag{3.28}$$

The proof that it is conserved is straightforward, we show directly that its time derivative vanishes,

$$\begin{aligned}
\dot{\boldsymbol{L}} &= \dot{\boldsymbol{r}}_1\times\boldsymbol{p}_1 + \boldsymbol{r}_1\times\dot{\boldsymbol{p}}_1 + \dot{\boldsymbol{r}}_2\times\boldsymbol{p}_2 + \boldsymbol{r}_2\times\dot{\boldsymbol{p}}_2 = 0 + \boldsymbol{r}_1\times\boldsymbol{F}_{12} + 0 + \boldsymbol{r}_2\times\boldsymbol{F}_{21} \\
&= (\boldsymbol{r}_1 - \boldsymbol{r}_2)\times\boldsymbol{F}_{12} = 0\ .
\end{aligned} \tag{3.29}$$

We use Newton's second law, that $\dot{\boldsymbol{p}}_1 = \boldsymbol{F}_{12}$ and $\dot{\boldsymbol{p}}_2 = \boldsymbol{F}_{21}$, Newton's third law, that $\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$, and the fact that, by Newton's law of gravitation, the force is a vector pointing along the vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$.

It is a rather natural guess that we may also split the total angular momentum as a sum of the angular momentum of the centre of mass, defined as

$$\boldsymbol{L}_{\text{cm}} = \boldsymbol{R}\times(M\boldsymbol{V})\ , \tag{3.30}$$

and the relative angular momentum, defined as

$$\boldsymbol{L}_{\text{rel}} = \boldsymbol{r}\times(m\boldsymbol{v})\ . \tag{3.31}$$

The proof that $\boldsymbol{L} = \boldsymbol{L}_{\text{cm}} + \boldsymbol{L}_{\text{rel}}$ is left as an exercise.

This decomposition leads to a second proof that $\boldsymbol{L}$ is conserved. In fact, we know that $\boldsymbol{L}_{\text{cm}}$ is conserved, because it is the angular momentum of a free particle. And we also know, from our study of the one-particle problem, that $\boldsymbol{L}_{\text{rel}}$ is conserved.

3.6 The virial theorem for two particles

The two-particle virial is

$$U = \boldsymbol{r}_1 \cdot \boldsymbol{p}_1 + \boldsymbol{r}_2 \cdot \boldsymbol{p}_2 \; . \tag{3.32}$$

And its time derivative is

$$\begin{aligned} \dot{U} &= \boldsymbol{v}_1 \cdot \boldsymbol{p}_1 + \boldsymbol{r}_1 \cdot \boldsymbol{F}_{12} + \boldsymbol{v}_2 \cdot \boldsymbol{p}_2 + \boldsymbol{r}_2 \cdot \boldsymbol{F}_{21} \\ &= m_1 \boldsymbol{v}_1^2 - \boldsymbol{r}_1 \cdot (\nabla_1 V) + m_2 \boldsymbol{v}_2^2 - \boldsymbol{r}_2 \cdot (\nabla_2 V) = 2E_K + V \; . \end{aligned} \tag{3.33}$$

Here E_K is the sum of the kinetic energies of the two particles, and $V = -GMm/r$ is the potential energy. In fact, we have that

$$\boldsymbol{r}_1 \cdot (\nabla_1 V) + \boldsymbol{r}_2 \cdot (\nabla_2 V) = (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \left(\frac{GMm}{r^3} \boldsymbol{r} \right) = \frac{GMm}{r} = -V \; . \tag{3.34}$$

Again, the relation

$$\boldsymbol{r}_1 \cdot (\nabla_1 V) + \boldsymbol{r}_2 \cdot (\nabla_2 V) = -V \tag{3.35}$$

follows from the general property of the potential energy

$$V(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}} \tag{3.36}$$

that it is a homogeneous function of degree -1. Scaling all six arguments $x_1, y_1, z_1, x_2, y_2, z_2$ by a common factor $\lambda > 0$ gives that

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2) = -\frac{GMm}{\lambda r} = \lambda^{-1} \, V(\boldsymbol{r}) \; . \tag{3.37}$$

Taking $\lambda = 1 + \epsilon$, where ϵ is infinitesimal, gives that

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2) = V(\boldsymbol{r}_1, \boldsymbol{r}_2) + \epsilon \, \boldsymbol{r}_1 \cdot \nabla_1 V(\boldsymbol{r}_1, \boldsymbol{r}_2) + \epsilon \, \boldsymbol{r}_2 \cdot \nabla_2 V(\boldsymbol{r}_1, \boldsymbol{r}_2) \; , \tag{3.38}$$

and that

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2) = (1 + \epsilon)^{-1} \, V(\boldsymbol{r}_1, \boldsymbol{r}_2) = (1 - \epsilon) \, V(\boldsymbol{r}_1, \boldsymbol{r}_2) \; . \tag{3.39}$$

Comparison of these two expressions gives Equation (3.35), which we wanted to prove.

The remaining part of the derivation of the virial theorem is the same as in the one-particle case. We take the time average of the equation $\dot{U} = 2E_K + V$, and get that

$$\frac{U(t_2) - U(t_1)}{t_2 - t_1} = 2\langle E_K \rangle + \langle V \rangle \; . \tag{3.40}$$

Again the left hand side of this equation vanishes if it includes one or more complete periods of a periodic orbit, or if the system is bound and we take the limit of an infinitely long time interval. Then we have that

$$2\langle E_K \rangle + \langle V \rangle = 0 \; . \tag{3.41}$$

And, because the total energy $E = E_K + V$ is constant,

$$E = \langle E_K \rangle + \langle V \rangle = -\langle E_K \rangle = \frac{\langle V \rangle}{2} \; . \tag{3.42}$$

Chapter 4

The many-particle problem

The gravitational many-particle problem can not be solved explicitly in a similar way as the two-particle problem. But we may formulate the problem, and prove useful general results, such as the conservation of energy and angular momentum, and the virial theorem.

4.1 The equations of motion

We assume now that there are N pointlike masses $m_1, m_2, \ldots, m_N$ acting on each other by gravitational forces. Their time dependent positions are $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$. Consider one of the particles, say particle number i. The total force acting upon it, $\boldsymbol{F}_i$, is the sum of the gravitational forces from all the other particles,

$$\boldsymbol{F}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \boldsymbol{F}_{ij} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \left(-\frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} (\boldsymbol{r}_i - \boldsymbol{r}_j) \right) . \tag{4.1}$$

We assume that there are no forces apart from the gravitational forces, then Newton's second law applied to particle i gives that

$$\frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = \frac{\boldsymbol{F}_i}{m_i} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \left(-\frac{G m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} (\boldsymbol{r}_i - \boldsymbol{r}_j) \right) . \tag{4.2}$$

4.2 The centre of mass and conservation of momentum

By definition, the position of the centre of mass of the N particles is

$$\boldsymbol{R} = \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M} \sum_{i=1}^{N} m_i \boldsymbol{r}_i , \tag{4.3}$$

where M is the sum of all the masses. The centre of mass moves with the velocity

$$\boldsymbol{V} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \frac{1}{M} \sum_{i=1}^{N} m_i \boldsymbol{v}_i = \frac{\boldsymbol{P}}{M} , \tag{4.4}$$

where $\boldsymbol{P}$ is the total momentum,

$$\boldsymbol{P} = \sum_{i=1}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} m_i \boldsymbol{v}_i \ . \tag{4.5}$$

The total momentum is conserved also in the many-particle case, because of Newton's second and third laws,

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \sum_{i=1}^{N} \frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t} = \sum_{i=1}^{N} \boldsymbol{F}_i = 0 \ . \tag{4.6}$$

The sum of all forces must vanish, as a consequence of Newton's third law. Here is a more formal proof. We have that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{F}_{ij} \ . \tag{4.7}$$

There are various ways to manipulate this double sum. First let us interchange the order of the summations, it gives that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \boldsymbol{F}_{ij} \ . \tag{4.8}$$

Then we rename the summation indices, this is allowed, because the value of a sum does not depend on the name of the summation index. Interchanging the names i and j we get that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{F}_{ji} \ . \tag{4.9}$$

It follows that

$$\sum_{i=1}^{N} \boldsymbol{F}_i = \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} (\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji}) = 0 \ , \tag{4.10}$$

where the last equality follows from Newton's third law.

The conservation of total momentum means that the centre of mass moves with a constant velocity.

4.3 Conservation of energy

The gravitational force $\boldsymbol{F}_i$ on particle i from all the other particles may be obtained as a negative gradient, $\boldsymbol{F}_i = -\nabla_i V$, of a total potential energy V. The correct definition of V should be clear from our study of the two-particle system. In fact, the potential energy of one pair of particles, say the particles number i and j, is the same as before,

$$V_{ij} = V_{ij}(\boldsymbol{r}_i, \boldsymbol{r}j) = -\frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}j|} \ . \tag{4.11}$$

And the total potential energy is a sum over all the particle pairs,

$$V = V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} V_{ij} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(-\frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r} j|} \right). \tag{4.12}$$

The proof that $\boldsymbol{F}_i = -\nabla_i V$ is left as an exercise, it is essentially a repetition of what we did in the two-particle case.

Obviously, the total kinetic energy is the sum of the kinetic energies of all the particles,

$$E_K = \sum_{i=1}^{N} \frac{1}{2} m_i \boldsymbol{v}_i^{\,2} . \tag{4.13}$$

And the total energy is

$$E = E_K + V = \sum_{i=1}^{N} \frac{1}{2} m_i \boldsymbol{v}_i^{\,2} - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{G m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r} j|} . \tag{4.14}$$

To prove that it is conserved, we have to show that its time derivative vanishes, that $\dot{E} = 0$. In a similar way as in the two-particle case, we have that

$$\dot{E} \;=\; \dot{E}_K + \dot{V} = \sum_{i=1}^{N} \left(m_i \boldsymbol{a}_i \cdot \boldsymbol{v}_i + (\nabla_i V) \cdot \dot{\boldsymbol{r}}_i \right) = \sum_{i=1}^{N} \left(m_i \boldsymbol{a}_i + \nabla_i V \right) \cdot \boldsymbol{v}_i = 0 . \tag{4.15}$$

4.4 Conservation of angular momentum

The total angular momentum is the sum of the angular momenta of the N particles,

$$\boldsymbol{L} = \sum_{i=1}^{N} \boldsymbol{L}_i = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{p}_i . \tag{4.16}$$

It is conserved, since its time derivative vanishes. In fact, we have that

$$\dot{\boldsymbol{L}} = \sum_{i=1}^{N} \boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_i = \sum_{i=1}^{N} \boldsymbol{r}_i \times \sum_{\substack{j=1 \\ j\neq i}}^{N} \boldsymbol{F}_{ij} = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} . \tag{4.17}$$

Newton's third law implies that

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji} = \frac{1}{2} \left(\boldsymbol{F}_{ij} - \boldsymbol{F}_{ji} \right) , \tag{4.18}$$

and hence,

$$\dot{\boldsymbol{L}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \left(\boldsymbol{r}_i \times \boldsymbol{F}_{ij} - \boldsymbol{r}_i \times \boldsymbol{F}_{ji} \right) . \tag{4.19}$$

By the old trick of changing the order of summations and the names of summation variables we get that

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} = \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i\neq j}}^{N} \boldsymbol{r}_i \times \boldsymbol{F}_{ji} = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} \boldsymbol{r}_j \times \boldsymbol{F}_{ij} , \tag{4.20}$$

and hence,

$$\dot{\boldsymbol{L}} = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}(\boldsymbol{r}_i - \boldsymbol{r}_j)\times \boldsymbol{F}_{ij} = 0\ . \tag{4.21}$$

Where we have used that the two-body force $\boldsymbol{F}_{ij}$ is central, that is, it points along the vector $\boldsymbol{r}_i - \boldsymbol{r}_j$.

Also in the N-particle case we may split the total angular momentum as

$$\boldsymbol{L} = \boldsymbol{L}_{\text{cm}} + \boldsymbol{L}_{\text{rel}}\ , \tag{4.22}$$

defining the angular momentum of the centre of mass as

$$\boldsymbol{L}_{\text{cm}} = \boldsymbol{R}\times\boldsymbol{P} = \boldsymbol{R}\times(M\boldsymbol{V})\ , \tag{4.23}$$

and then defining the relative angular momentum as

$$\boldsymbol{L}_{\text{rel}} = \boldsymbol{L} - \boldsymbol{L}_{\text{cm}}\ . \tag{4.24}$$

In the absence of external forces, $\boldsymbol{L}_{\text{cm}}$ is conserved, because it is the angular momentum of a free particle. Since the total angular momentum $\boldsymbol{L}$ is conserved, it follows that the relative angular momentum $\boldsymbol{L}_{\text{cm}}$ is conserved.

4.5 The virial theorem for N particles

The N-particle virial is defined by the obvious generalization as

$$U = \sum_{i=1}^{N}\boldsymbol{r}_i\cdot\boldsymbol{p}_i\ . \tag{4.25}$$

Its time derivative is

$$\dot{U}\ = \sum_{i=1}^{N}(\boldsymbol{v}_i\cdot\boldsymbol{p}_i + \boldsymbol{r}_i\cdot\boldsymbol{F}_i) = \sum_{i=1}^{N}\Big(m_i{\boldsymbol{v}_i}^2 - \boldsymbol{r}_i\cdot(\nabla_i V)\Big) = 2E_K + V\ . \tag{4.26}$$

Here E_K is the sum of the kinetic energies of the N particles, and V is the total potential energy, as defined in Equation (4.12).

Like before, in the one- and two-particle cases, the relation

$$\sum_{i=1}^{N}\boldsymbol{r}_i\cdot(\nabla_i V) = -V \tag{4.27}$$

may be proved directly, or we may use that the potential energy V is a homogeneous function of degree -1,

$$V(\lambda\boldsymbol{r}_1,\lambda\boldsymbol{r}_2,\ldots,\lambda\boldsymbol{r}_N) = \lambda^{-1}\,V(\boldsymbol{r},\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N)\ . \tag{4.28}$$

Taking $\lambda = 1+\epsilon$, where ϵ is infinitesimal, gives that

$$V(\lambda\boldsymbol{r}_1,\lambda\boldsymbol{r}_2,\ldots,\lambda\boldsymbol{r}_N) = V(\boldsymbol{r},\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N) + \epsilon\sum_{i=1}^{N}\boldsymbol{r}_i\cdot\nabla_i V(\boldsymbol{r},\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N)\ , \tag{4.29}$$

and that

$$V(\lambda \boldsymbol{r}_1, \lambda \boldsymbol{r}_2, \ldots, \lambda \boldsymbol{r}_N) = (1-\epsilon)\, V(\boldsymbol{r}, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N)\ . \tag{4.30}$$

Comparison of these two expressions gives Equation (4.27).

Exactly as before, we conclude that if we take the time average over one or more complete periods of a periodic motion, or if the system is bound and we take the limit of an infinitely long time interval, then

$$2\langle E_K\rangle + \langle V\rangle = 0\ . \tag{4.31}$$

And, because the total energy $E = E_K + V$ is constant,

$$E = \langle E_K\rangle + \langle V\rangle = -\langle E_K\rangle = \frac{\langle V\rangle}{2}\ . \tag{4.32}$$

4.6 The central temperature of the Sun

The virial theorem has a direct application in understanding the stability of the Sun. The material in the Sun is a gas, consisting mostly of free electrons, protons and helium nuclei. It can be reasonably well described by the ideal gas law,

$$P\mathcal{V} = N k_B T\ , \tag{4.33}$$

where P is the pressure, $\mathcal{V}$ the volume of the gas, N the number of gas particles, T the temperature, and $k_B = 1.38\ 10^{-23}$ J/K is Boltzmann's constant. The temperature and the density vary of course much from the centre of the Sun and out to the surface, but we may still in a meaningful way talk about the average temperature, which will be just a little bit lower than the temperature at the centre, where most of the mass is concentrated.

The Sun is a very stable system, it has existed already some 4.5 10^9 years in essentially the same state. This can happen because it is gravitationally bound, and there exists a stable equilibrium between the gravitational force, directed inwards, and the pressure, directed outwards. The pressure is essentially that of an ideal gas, which is entirely due to the kinetic energy of the gas particles. The average kinetic energy of one particle is

$$E_{\text{av}} = \frac{3}{2}\, k_B T\ . \tag{4.34}$$

When energy is radiated away from the surface of the Sun, the internal pressure is maintained by conversion of nuclear energy into kinetic energy.

The stability of the Sun means that the virial theorem applies. It tells us that the total kinetic energy is minus half of the total gravitational potential energy,

$$N E_{\text{av}} = -\frac{V}{2} \approx \frac{G M_\odot^2}{2 R_\odot}\ . \tag{4.35}$$

Here $M_\odot = 2.0\ 10^{30}$ kg is the mass and $R_\odot = 7.0\ 10^8$ m the radius of the Sun. This expression for the potential energy is only approximately valid. The exact expression includes another numerical factor, which would be 3/5 for a sphere of uniform density, and must be larger than 3/5 for the Sun, which has much higher density at its centre than at the surface. It seems a reasonable guess to take the numerical factor to be one.

Thus, we estimate the average temperature of the Sun to be

$$T \approx \frac{GM_\odot^2}{3Nk_BR_\odot} = \frac{GM_\odot\overline{m}}{3k_BR_\odot} \ . \tag{4.36}$$

The ratio $M_\odot/N = \overline{m}$ is the average mass of the gas particles. The gas consists mainly of hydrogen and helium, about 3/4 of the mass is hydrogen, and 1/4 is helium. This means that, on the average, for every 12 free protons, each of mass $m_p = 1.67\ 10^{-27}$ kg, we have one helium nucleus, of mass approximately $4m_p$, and 14 electrons, each of mass $m_e = 9.1\ 10^{-31}$ kg, altogether 27 particles. It follows that the average particle mass is

$$\overline{m} \approx \frac{16m_p}{27} \ . \tag{4.37}$$

Hence,

$$T \approx \frac{16GM_\odot m_p}{81k_BR_\odot} = \frac{16\cdot 6.67\ 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}\ \ 2.0\ 10^{30}\ \mathrm{kg}\ 1.67\ 10^{-27}\ \mathrm{kg}}{81\cdot 1.38\ 10^{-23}\ \mathrm{J/K}\ 7.0\ 10^{8}\ \mathrm{m}} = 4.6\ 10^6\ \mathrm{K} \ . \tag{4.38}$$

Remember that this is an estimate of the average temperature of the Sun, and the temperature at the centre must be somewhat higher. It is actually about three times as high, according to the "standard solar model". But we see that the estimate we arrived at gives a rather good idea about the central temperature of the Sun. It must be around ten million kelvin, it could not be, for example, one million kelvin, or one hundred million kelvin.

4.7 Gravitational stability of a gas cloud

A spherical gas cloud of mass M and radius R, of uniform mass density ρ (mass per volume) and uniform temperature, has a critical temperature T_J, called the *Jeans temperature*, which is given by equation (4.36) with an extra factor of 3/5 for the exact gravitational energy,

$$T_J = \frac{GM^2}{5Nk_BR} = \frac{GM\overline{m}}{5k_BR} \ . \tag{4.39}$$

As before, N is the number of gas particles, and $\overline{m} = M/N$ is the average particle mass.

If the temperature T is equal to the Jeans temperature T_J, then the time derivative of the virial vanishes,

$$\dot{U} = 2E_K + V = E + E_K = 2E - V = 0 \ . \tag{4.40}$$

We found that this relation is a necessary condition for gravitational equilibrium. However, it is only a *global* equilibrium condition. In addition to the global condition, there must hold everywhere a *local* equilibrium condition, that the pressure increases towards the centre of the cloud in such a way that it balances the gravitational attraction between all parts of the cloud. When the pressure increases towards the centre, the density must also increase.

If we take it for granted that the cloud is going to reach a state where it is in gravitational equilibrium, then the virial theorem tells us something about what the equilibrium state will be. This is so, because the total energy E is a constant of motion, unless energy is supplied to or removed from the cloud by interaction with the enironment. One important mechanism for removing energy is thermal radiation. However, it takes time to radiate away a substantial amount of energy, and on a shorter time scale the energy E is constant, to a good approximation.

The virial theorem states that the relation

$$\dot{U} = 2E - V = 0 \tag{4.41}$$

must hold in equilibrium. Thus, if $T = T_J$, so that $\dot{U} = 0$ to begin with, and if E is constant, then the cloud in its future equilibrium state must have exactly the same gravitational potential energy V as it starts out with. If V can not change, it means essentially that the cloud can neither contract nor expand.

The only way that the cloud is able to contract slowly, on a longer time scale, while maintaining its gravitational equilibrium on the shorter time scale, is that it radiates slowly away its energy E. The paradoxical result when the energy E is reduced, which means that the absolute value $|E|$ increases, is that the absolute value $|V| = 2\,|E|$ of the potential energy V increases, as a result of contraction of the cloud, and that the kinetic energy $E_K = |V|/2 = |E|$ also increases. Thus, a gas in gravitational equilibrium has the strange property that when it radiates away energy, its temperature increases! Unless there exists a mechanism for converting nuclear energy, or some other form of energy, into kinetic energy, as there exists inside the Sun.

If the cloud has a temperature below its Jeans temperature, that is, if $T < T_J$, then it must contract as a whole due to internal gravitational forces. This must be so, because the gas pressure is too low to resist the contraction. That the pressure is too low, means that the gas particles have too little kinetic energy compared to the potential energy, so that the time derivative of the virial is negative,

$$\dot{U} = 2E_K + V = E + E_K = 2E - V < 0\,. \tag{4.42}$$

If the energy E is constant, the only way to establish the global equilibrium condition $\dot{U} = 2E - V = 0$, is to make the potential more negative, by contraction. Thus, the cloud will be unstable against contraction if $T < T_J$.

The condition for instability, $T < T_J$, may be written in several equivalent ways. Expressing the mass in terms of the density and radius,

$$M = \frac{4\pi}{3}\,\rho R^3\,, \tag{4.43}$$

we may write the instability condition as $R > R_J$, where R_J is a critical radius, the *Jeans radius*,

$$R_J = \sqrt{\frac{15 k_B T}{4\pi G \rho\,\overline{m}}}\,. \tag{4.44}$$

Or we write it as $M > M_J$, where M_J is the *Jeans mass*,

$$M_J = \frac{4\pi}{3}\,\rho R_J^3 = \sqrt{\frac{3}{4\pi\rho}\left(\frac{5 k_B T}{G\,\overline{m}}\right)^3}\,. \tag{4.45}$$

Appendix A

Units, vectors, and other notation

A.1 Units

A general rule is that if a physical unit is the name of a person, then it is either written abbreviated as a capital (or with the first letter in capital if the abbreviation has more than one letter), or written in full with no capital. Examples: N = newton, J = joule, C = coulomb, A = ampere. There is no plural "s" ending in unit names: 10 second, not 10 seconds; 6 newton, not 6 newtons.

The unit of time, the *second*, s, is defined by the frequency of electromagnetic waves that are in resonance with the hyperfine transition in cesium atoms. This resonance frequency is by definition

$$\nu_0 = 9\,192\,631\,770\ \mathrm{s}^{-1} = 9\,192\,631\,770\ \mathrm{Hz}\ . \tag{A.1}$$

This is a good operational definition, because it is possible to build an electric circuit oscillating with this frequenecy, and tune it very accurately to be in resonance with cesium atoms. Then one measures time by counting oscillations.

The speed of light in vacuum,

$$c = 299\,792\,458\ \mathrm{m/s}\ . \tag{A.2}$$

is exact because it defines the *meter*, m, as the unit of length.

A.2 Basic notation

This section is a summary of some standard notation.

Time is usually denoted by the symbol t.

A position in three dimensional space is specified by three coordinates (x, y, z), measured along three orthogonal axes. The coordinates have the dimension of length.

Vectors are written here in boldface, but when writing by hand it is easier to use vector arrows. Thus, $\boldsymbol{A}$ and $\vec{A}$ are two notations for the same vector. A vector in general may have any number of components, but our vectors will usually have three components. The components of the three dimensional vector $\boldsymbol{A}$ are called (A_x, A_y, A_z).

The unit vectors along the three orthogonal (x, y, z) coordinate axes are called $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, or in handwriting $\vec{i}$, $\vec{j}$, $\vec{k}$. In general, a vector $\boldsymbol{A}$ is a linear combination of the basis vectors,

$$\boldsymbol{A} = A_x\,\boldsymbol{i} + A_y\,\boldsymbol{j} + A_z\,\boldsymbol{k}\ . \tag{A.3}$$

The point in space with coordinates (x, y, z) is represented as a position vector

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\ . \tag{A.4}$$

If $\boldsymbol{r}$ is the time dependent position of a pointlike particle, $\boldsymbol{r} = \boldsymbol{r}(t)$, then the first and second time derivatives of the position are respectively the velocity $\boldsymbol{v}$ and the acceleration $\boldsymbol{a}$,

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\ , \qquad \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\ . \tag{A.5}$$

It is often convenient to denote a time derivative by a dot, thus we write

$$\boldsymbol{v} = \dot{\boldsymbol{r}}\ , \qquad \boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}\ . \tag{A.6}$$

All of these equations are vector equations. The equation

$$\boldsymbol{v} = v_x\,\boldsymbol{i} + v_y\,\boldsymbol{j} + v_z\,\boldsymbol{k} = \dot{\boldsymbol{r}} = \dot{x}\,\boldsymbol{i} + \dot{y}\,\boldsymbol{j} + \dot{z}\,\boldsymbol{k}\ , \tag{A.7}$$

for example, consists of the three separate equations

$$v_x = \dot{x}\ , \qquad v_y = \dot{y}\ , \qquad v_z = \dot{z}\ . \tag{A.8}$$

Scalar and vector products of vectors

The *scalar product* (dot product, inner product) of two vectors $\boldsymbol{A} = A_x\,\boldsymbol{i} + A_y\,\boldsymbol{j} + A_z\,\boldsymbol{k}$ and $\boldsymbol{B} = B_x\,\boldsymbol{i} + B_y\,\boldsymbol{j} + B_z\,\boldsymbol{k}$ is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z\ . \tag{A.9}$$

The scalar product is symmetric, or commutative, $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$. The length of the vector $\boldsymbol{A}$ is written as $|\boldsymbol{A}|$, or A, and the length squared is

$$A^2 = |\boldsymbol{A}|^2 = \boldsymbol{A}^2 = \boldsymbol{A} \cdot \boldsymbol{A} = A_x^2 + A_y^2 + A_z^2\ . \tag{A.10}$$

Two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are *orthgonal* if $\boldsymbol{A} \cdot \boldsymbol{B} = 0$. The vector $\boldsymbol{A}$ is a *unit* vector, or *normal* vector, if it has unit length, $|\boldsymbol{A}| = 1$. Two or more vectors are *orthonormal* if they are all unit vectors, and if any two of them are orthogonal.

The *vector product* (cross product, outer product) of the same two vectors is defined as

$$\boldsymbol{A} \times \boldsymbol{B} = (A_yB_z - A_zB_y)\,\boldsymbol{i} + (A_zB_x - A_xB_z)\,\boldsymbol{j} + (A_xB_y - A_yB_x)\,\boldsymbol{k}\ . \tag{A.11}$$

The vector product is antisymmetric, or anticommutative, $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$. In particular, the antisymmetry relation $\boldsymbol{A} \times \boldsymbol{A} = -\boldsymbol{A} \times \boldsymbol{A}$ has the unique solution $\boldsymbol{A} \times \boldsymbol{A} = 0$: the vector product of a vector with itself vanishes.

Note that the vector product is not associative, unlike other products we are used to. For example, $\boldsymbol{i} \times (\boldsymbol{i} \times \boldsymbol{j}) = \boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j} \neq (\boldsymbol{i} \times \boldsymbol{i}) \times \boldsymbol{j} = 0 \times \boldsymbol{j} = 0$. Thus we have in general that

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) \neq (\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C}\ . \tag{A.12}$$

The scalar product is also not associative, for the simple reason that an expression like $\boldsymbol{A} \cdot (\boldsymbol{B} \cdot \boldsymbol{C})$ is meaningless. The scalar product $\boldsymbol{B} \cdot \boldsymbol{C}$ of the two vectors $\boldsymbol{B}$ and $\boldsymbol{C}$ is a scalar, not a vector, and a scalar product between a vector and a scalar, such as $\boldsymbol{A}\cdot(\boldsymbol{B}\cdot\boldsymbol{C})$, has no meaning.

The *triple product*

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x) \tag{A.13}$$

is completely antisymmetric, for example, $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = -\boldsymbol{A}\cdot(\boldsymbol{C}\times\boldsymbol{B})$, and

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C} = -(\boldsymbol{B}\times\boldsymbol{A})\cdot\boldsymbol{C} = -\boldsymbol{B}\cdot(\boldsymbol{A}\times\boldsymbol{C})\ . \tag{A.14}$$

Hence, $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = 0$ whenever two of the three vectors are either equal or proportional. The triple product may also be defined as a determinant,

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \begin{vmatrix} A_x & B_x & C_x \\ A_y & B_y & C_y \\ A_z & B_z & C_z \end{vmatrix}\ . \tag{A.15}$$

The geometrical interpretation of the scalar product is that $\boldsymbol{A}\cdot\boldsymbol{B} = AB\cos\alpha$, where α is the angle between the two vectors. The length of the vector product, in terms of the same angle, is

$$|\boldsymbol{A}\times\boldsymbol{B}| = AB\sin\alpha\ . \tag{A.16}$$

This is the area of the parallelogram spanned by the two vectors, that is, with the two vectors as two of its sides. The triple product is the (positive or negative) volume of the parallelepiped spanned by the three vectors.

A.3 Scalar and vector fields

A *scalar* is either one number, or a physical quantity such that when it is measured, the result will be one number times a physical unit. The important point is that a scalar has only one component, as opposed to a vector, which has (usually) three components.

A *scalar field* ϕ is a function having a scalar value $\phi(x,y,z,t) = \phi(\boldsymbol{r},t)$ at any given position $\boldsymbol{r}$ at any given time t. Similarly, a *vector field* $\boldsymbol{A}$ is a function having a vector value $\boldsymbol{A}(x,y,z,t) = \boldsymbol{A}(\boldsymbol{r},t)$ at the position $\boldsymbol{r}$ at the time t. Very often, when we speak of a scalar or a vector, we actually mean a scalar field or a vector field.

As an example from meteorology, the temperature distribution in the atmosphere is a scalar field, whereas the distribution of wind velocities is a vector field.

A.4 Differentiation

If $f = f(x)$ is a function of one variable x, then the derivative of f with respect to x is defined as

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\ . \tag{A.17}$$

We also write

$$f' = \frac{\mathrm{d}f}{\mathrm{d}x}\ . \tag{A.18}$$

A third notation for differentiation is the dot for the time derivative, as introduced above.

We write differentiation with respect to x as $\mathrm{d}/\mathrm{d}x$ when x is the only variable. If $f = f(x,y,z)$ is a function of the three variables x, y, z, then we write the *partial differentiation* with respect to one variable x, for fixed values of y and z, as

$$\frac{\partial f(x,y,z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y,z) - f(x,y,z)}{\Delta x} \ . \tag{A.19}$$

The three partial derivatives of f with respect to x, y and z may be regarded as the components of a vector, the *gradient* of f, defined as

$$\nabla f = \frac{\partial f}{\partial x}\,\boldsymbol{i} + \frac{\partial f}{\partial y}\,\boldsymbol{j} + \frac{\partial f}{\partial z}\,\boldsymbol{k} \ . \tag{A.20}$$

The gradient vector at one point is orthogonal to the level curve of f (the curve along which f is constant) going through this point. It points in the direction in which f is increasing the fastest, and its length is the rate of increase of f in this direction.

We may think of ∇ (called "nabla", or "del") as an operator,

$$\nabla = \boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z} \ , \tag{A.21}$$

producing a vector field ∇f when it acts on a scalar field f.

The chain rule

Assume, for example, that f is a function of one variable x, which in turn is a function of t, so that f is also a function of t. The chain rule tells us how to differentiate $f = f(x(t))$ with respect to t,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t} \ . \tag{A.22}$$

If g is a function of three variables x, y, x, all of which are functions of t, so that g is in the end a function of the single variable t, then the t derivative of $g = g(x(t), y(t), z(t))$ is

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial g}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}t} \ . \tag{A.23}$$

In this formula we write the time derivatives as $\mathrm{d}/\mathrm{d}t$, because we differentiate functions of one single variable t, whereas we write the x derivative as $\partial/\partial x$, because it applies to a function depending not only on x, but also on two other variables y and z. The first kind of derivative, applying to functions of one variable, is called a *total* derivative, and the second kind, applying to functions of several variables, is called a *partial* derivative.

In vector notation we write the same formula as above, i.e. the chain rule for the function $g = g(\boldsymbol{r}(t))$, in the following way,

$$\frac{\mathrm{d}g}{\mathrm{d}t} = (\nabla g)\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\nabla g \ . \tag{A.24}$$

If g is a function of the four variables x, y, z, t, and if all four of these are functions of a fifth variable u, so that $g = g(x(u), y(u), z(u), t(u))$, then by the same chain rule as above we have that

$$\frac{\mathrm{d}g}{\mathrm{d}u} = \frac{\partial g}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}u} + \frac{\partial g}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}u} + \frac{\partial g}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}u} + \frac{\partial g}{\partial t}\,\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\nabla g + \frac{\partial g}{\partial t}\,\frac{\mathrm{d}t}{\mathrm{d}u} \ . \tag{A.25}$$

In the specaial case $t = u$ we will have that

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial g}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial g}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \cdot \nabla g + \frac{\partial g}{\partial t} . \tag{A.26}$$

In this case there is an important difference between the partial time derivative $\partial g/\partial t$, which applies to only the fourth argument of the function $g = g(x, y, z, t)$, and the total time derivative $\mathrm{d}g/\mathrm{d}t$, which applies also to the time dependence of the first three arguments x, y, z.