Casimir-effekten

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Se: Per Christian Hemmer “Kvantemekanikk”, s. 327–332.
Her er en enklere (og mindre suspekt?) utregning.
Elektromagnetisk felt i en boks med ledende vegger, sidekanter L_x, L_y, L_z.
Moder: tillatte bølgetall er

$$k_x = \frac{l\pi}{L_x}\,, \qquad k_y = \frac{m\pi}{L_y}\,, \qquad k_z = \frac{n\pi}{L_z}\,, \qquad k = \sqrt{{k_x}^2 + {k_y}^2 + {k_z}^2}\,, \tag{1}$$

med $l, m, n = 0, 1, 2, \ldots$.
To moder for $l > 0$, $m > 0$ og $n > 0$.
En mode dersom enten $l = 0$ eller $m = 0$ eller $n = 0$, ikke to samtidig.
Ingen moder dersom to av tallene l, m, n er 0 samtidig.
Vinkelfrekvens $\omega_k = ck$, vakuumenergi i hver mode

$$E_k = \frac{1}{2}\,\hbar\omega_k = \frac{\hbar c\pi}{2}\sqrt{\frac{l^2}{L_x^2} + \frac{m^2}{L_y^2} + \frac{n^2}{L_z^2}}\,. \tag{2}$$

Den totale vakuumenergien, E_k summert over alle moder, er høyst divergent. Men hvis vi lar oss inspirere av statistisk mekanikk og velger en $\beta > 0$, kan vi definere en sum som er konvergent,

$$E(\beta) = \sum_{\text{alle moder}} E_k\, \mathrm{e}^{-\beta E_k} = -\frac{\mathrm{d}}{\mathrm{d}\beta}\, Z(\beta)\,, \tag{3}$$

der vi altså innfører en “partisjonsfunksjon”

$$Z(\beta) = \sum_{\text{alle moder}} \mathrm{e}^{-\beta E_k}\,. \tag{4}$$

Vi antar at i hvert fall de to sidekantene L_x og L_y er så store at vi kan integrere over l og m i stedet for å summere. Vi har da at

$$Z(\beta) = f(0) + 2\sum_{n=1}^{\infty} f(n)\,, \tag{5}$$

der

$$f(n) = \int_0^\infty \mathrm{d}l \int_0^\infty \mathrm{d}m\ \mathrm{e}^{-\beta E_k} = L_x L_y \int_0^\infty \mathrm{d}l' \int_0^\infty \mathrm{d}m'\ \mathrm{e}^{-\beta E_k}\,, \tag{6}$$

med $l = L_x l'$ og $m = L_y m'$. Definer

$$s^2 = (l')^2 + (m')^2 = \frac{l^2}{L_x^2} + \frac{m^2}{L_y^2} = \frac{4E_k^2}{(\hbar c\pi)^2} - \frac{n^2}{L_z^2}\,. \tag{7}$$

Da har vi at

$$f(n) = L_x L_y \, \frac{\pi}{2} \int_0^\infty s \, \mathrm{d}s \; \mathrm{e}^{-\beta E_k} = \frac{\pi L_x L_y}{2} \, \frac{4}{(\hbar c \pi)^2} \int_{E_{kn}}^\infty E_k \, \mathrm{d}E_k \; \mathrm{e}^{-\beta E_k} \; , \tag{8}$$

med

$$E_{kn} = \frac{\hbar c \pi n}{2 L_z} = n E_{k1} \; . \tag{9}$$

Vi har:

$$f(n) = -\frac{2 L_x L_y}{\pi (\hbar c)^2} \, \frac{\mathrm{d}}{\mathrm{d}\beta} \int_{E_{kn}}^\infty \mathrm{d}E_k \; \mathrm{e}^{-\beta E_k} = -\frac{2 L_x L_y}{\pi (\hbar c)^2} \, \frac{\mathrm{d}}{\mathrm{d}\beta} \left(\frac{\mathrm{e}^{-\beta E_{kn}}}{\beta} \right) \; . \tag{10}$$

Og videre:

$$Z(\beta) = -\frac{2 L_x L_y}{\pi (\hbar c)^2} \, \frac{\mathrm{d}}{\mathrm{d}\beta} \left(\frac{1}{\beta} + \frac{2}{\beta} \sum_{n=1}^\infty \mathrm{e}^{-\beta n E_{k1}} \right) = -\frac{2 L_x L_y}{\pi (\hbar c)^2} \, \frac{\mathrm{d}}{\mathrm{d}\beta} \left(\frac{1}{\beta} + \frac{2}{\beta} \frac{\mathrm{e}^{-\beta E_{k1}}}{1 - \mathrm{e}^{-\beta E_{k1}}} \right) \; . \tag{11}$$

Vi bruker definisjonen på Bernoulli-tallene B_n,

$$\frac{\mathrm{e}^{-\beta E_{k1}}}{1 - \mathrm{e}^{-\beta E_{k1}}} = \frac{1}{\mathrm{e}^{\beta E_{k1}} - 1} = \sum_{n=0}^\infty B_n \, \frac{(\beta E_{k1})^{n-1}}{n!} = \frac{1}{\beta E_{k1}} - \frac{1}{2} + \frac{\beta E_{k1}}{12} - \frac{(\beta E_{k1})^3}{720} + \mathcal{O}(\beta^4) \; . \tag{12}$$

Det gir at

$$\begin{aligned}
Z(\beta) &= -\frac{2 L_x L_y}{\pi (\hbar c)^2} \, \frac{\mathrm{d}}{\mathrm{d}\beta} \left(\frac{2}{\beta^2 E_{k1}} + \frac{E_{k1}}{6} - \frac{\beta^2 E_{k1}^3}{360} + \mathcal{O}(\beta^3) \right) \\
&= \frac{8 L_x L_y}{\pi (\hbar c)^2 \beta^3 E_{k1}} + \frac{L_x L_y \beta E_{k1}^3}{90 \pi (\hbar c)^2} + \mathcal{O}(\beta^2) \\
&= \frac{16 L_x L_y L_z}{\pi^2 (\hbar c)^3 \beta^3} + \frac{\pi^2 L_x L_y \hbar c \beta}{720 L_z^3} + \mathcal{O}(\beta^2) \; .
\end{aligned} \tag{13}$$

Den regulariserte vakuumenergien blir da

$$E(\beta) = -\frac{\mathrm{d}}{\mathrm{d}\beta} \, Z(\beta) = \frac{48 L_x L_y L_z}{\pi^2 (\hbar c)^3 \beta^4} - \frac{\pi^2 L_x L_y \hbar c}{720 L_z^3} + \mathcal{O}(\beta) \; . \tag{14}$$

Eller, uttrykt ved $h = 2\pi\hbar$,

$$E(\beta) = \frac{384 \pi L_x L_y L_z}{(hc)^3 \beta^4} - \frac{\pi L_x L_y hc}{1440 L_z^3} + \mathcal{O}(\beta) \; . \tag{15}$$