NORGES TEKNISK-NATURVITENSKAPELIGE UNIVERSITET
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Faglig kontakt under eksamen:

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EXAM IN SIF4088 NONLINEAR DYNAMICS

Wednesday 18. December 2002
09.00 – 15.00

Tillatte hjelpemidler: Godkjent kalkulator
Rottmann: *Matematisk formelsamling*
Barnett and Cronin: *Mathematical Formulae*

Results will be announced on 8. January 2003.

Problem 1

a) What is a permanent wave, a solitary wave and a soliton? Sketch different types of solitons, with one-soliton solutions of the Korteweg-de Vries equation, the cubic Schrödinger equation and the sine-Gordon equation as examples.

b) Let $u(x,t)$ satisfy the Korteweg-de Vries equation in the dimensionless form

$$u_t + u\,u_x + u_{xxx} = 0. \tag{1}$$

Show that for a localised pulse

$$C_1 = \int_{-\infty}^{\infty} u(x,t)\,dx$$

and

$$C_2 = \int_{\infty}^{\infty} \tfrac{1}{2}u^2\,dx$$

are constants of motion.

c) Show that the center of gravity

$$X(t) = \frac{\int_{-\infty}^{\infty} x\, u(x,t)\, dx}{\int_{-\infty}^{\infty} u(x,t)\, dx}$$

for a localised pulse move with constant velocity.

d) The Korteweg-de Vries equation describes, among other things, surface waves when an incompressible nonviscous fluid flows without vortices. Under what physical conditions is this a good description? (Answer briefly.)

Problem 2

In this problem we study the iteration

$$x_{n+1} = 1 - \mu\, x_n^2, \tag{2}$$

with μ as control parameter, and with a start value x_1 satisfying $|x_1| < 1$. We assume that $0 < \mu < 2$.

a) Determine the fixed points of the iteration. What is the largest value of the control parameter, μ_1, that corresponds to a stable fixed point? (Give your reasons for the answer.)

b) The attractor for iterations with period two is two values, x_+ og x_-, between which the iteration oscillates. Show first that

$$x_+ + x_- = 1/\mu,$$

and use this to show that

$$x_\pm = \frac{1 \pm \sqrt{4\mu - 3}}{2\mu}.$$

c) What is the largest value, μ_2, of the control parameter corresponding to a *stable* period-two iteration? For which value of μ is the period-two iteration superstable?

d) What does the Lyapunov exponent λ for a general one-dimensional iteration $x_{n+1} = F(x_n)$ describe?

Which sign of the Lyapunov exponent for the iteration (2) do you expect, when the control parameter satisfies $\mu_1 < \mu < \mu_2$, where μ_1 og μ_2 are defined above. Sketch qualitativly how the Lyapunov exponent varies as function of the control parameter in this interval? (For these questions no calculations are necessary.)

Problem 3

In a limited region live two competing species, A and B, competing, for instance, for the available food. The species have number densities at time t equal to $n_A(t)$ og $n_B(t)$, respectively. For very small number densities the reproduction rates are equal to r_A og r_B, respectively, but with increasing densities the populations increase slower. The time development is given by

$$\begin{aligned} \frac{dn_A}{dt} &= n_A\left(r_A - a_A n_A - b_A n_B\right) \\ \frac{dn_B}{dt} &= n_B\left(r_B - a_B n_B - b_B n_A\right). \end{aligned} \tag{3}$$

Here r_A, r_B, a_A, b_A, a_B and b_B are positive constants. The constants a_A and a_B describe how the population growth is inhibited due to competion between members of the same species, while the constants b_A and b_B describe how the population growth is inhibited due to competion with the other species.

a) Can the dynamical system (3) behave chaotically?

b) Assume first that the there is no competion between the species, i.e. $b_A = b_B = 0$. Explain qualitatively how the densities develop in time, from initial values $n_A(0) > 0$ and $n_B(0) > 0$, and give the precise values for $n_A(\infty)$ and $n_B(\infty)$ in this case.

c) Assume from now on that the inter-species competition cannot be neglected ($b_A \neq 0$ og $b_B \neq 0$), and assume for simplicity that $r_A = r_B = r$. Scale the densities and introduce a scaled time variable τ such that the equations (3) take the form

$$\begin{aligned} \frac{dx}{d\tau} &= x(1 - x - \alpha y) \\ \frac{dy}{d\tau} &= y(1 - y - \beta x), \end{aligned} \tag{4}$$

where x is proportional with n_A and y is proportional with n_B. What are the new constants (the control parameters) α og β in terms of the constants in (3)?

d) Find all fixed points for the dynamical system (4).

For one of the fixed points, $F_0 = (x^0, y^0)$, both x^0 and y^0 can be nonzero. For which values of the control parameters will this fixed point lie in the physical relevant quadrant of phase space, $x \geq 0, y \geq 0$?

e) Classify all physically interesting fixed points you have found in d) with respect to type (saddle point, stable/unstable node, stable/unstable spiral (focus), or center). Assume $\alpha \neq 1, \beta \neq 1$

f) Situations in which each species has some species-specific food available in addition to the food that the species compete for, corresponds to a dynamical system in which the control parameters α and β are small. Take as an example $\alpha = \beta = \frac{1}{2}$, and give a rough qualitative sketch of the phase portrait (the flow in the physical relevant part of phase space) in this case. (No limit cycles are present.) What are the dimensionless population densities x and y at $t = \infty$?

g) If the species are microorganisms that produce substances that are toxic for the competing species, then the presence of the other species will be of crucial importance, and this corresponds to large values of the control parameters α and β. As an example take $\alpha = \beta = 2$, and give a rough qualitative sketch of the phase portrait (in the physical relevant part of phase space) for this case. (Again no limit cycles are present.) What is the result for $t = \infty$ in this case?