The Norwegian University of Science and Technology
Department of Physics

Contact person:
Name: Jan Myrheim
Telephone: 93653

Examination, course TFY 4305 Nonlinear dynamics
Tuesday December 16, 2003
Time: 09.00–15.00

Grades made public: Tuesday January 6, 2004

Allowed to use: (Alternative C): Allowed pocket calculator.
Rottmann, *Mathematische Formelsammlung.*
Barnett and Cronin, *Mathematical Formulae.*
Øgrim og Lian, *Størrelser og enheter i fysikk og teknikk.*

Problem 1:

a) What is an autonomous dynamical system?
Are oscillations possible in a one dimensional autonomous system with continuous time?
In a one dimensional non-autonomous system with continuous time?
In a one dimensional autonomous system with discrete time?
Motivate your answers (briefly).

b) What happens in a Hopf bifurcation of a fixed point in a two dimensional dynamical system with continuous time?
Why is there an essential difference between a supercritical and a subcritical Hopf bifurcation?
What other types of bifurcations may occur, in this connection?
How is it possible to recognize a Hopf bifurcation by means of linear stability analysis?

c) Define what is meant by Lyapunov exponents (or Liapunov exponents), both in a dynamical system with continuous time, and in a system with discrete time.
What can you say about the three Lyapunov exponents in a three dimensional dynamical system with continuous time, if the motion is a stable limit cycle?
If the motion is a stable quasiperiodic orbit?

d) Define deterministic chaos.
Why is deterministic chaos impossible in a two dimensional autonomous system with continuous time?
Why is deterministic chaos nevertheless possible in a one dimensional autonomous system with discrete time (e.g. in the logistic mapping)?

Problem 2:

a) The one dimensional mapping

$$x \mapsto \widetilde{x} = f(x) = c + \epsilon x - x^2 \tag{1}$$

is one version of the logistic mapping. Here x is a real variable. Furthermore, c and ϵ are constant real parameters (control parameters), and we assume throughout this problem that ϵ is small, that is, $|\epsilon| \ll 1$.
Show that the mapping has two fixed points if

$$c > -\frac{(1-\epsilon)^2}{4} \, ,$$

and that one fixed point is always unstable, whereas the second one is stable for

$$c < 1 - \frac{(1-\epsilon)^2}{4} \, .$$

b) Assume now that one parameter ϵ is held constant, and that the second parameter c at first has a value such that one fixed point is stable. How does the system behave if the value of c is increased such that the fixed point becomes unstable?

c) In the remaining part of this problem we will study the behaviour of two logistic mappings that are weakly coupled to each other. Thus, we consider the two dimensional mapping

$$(x, y) \mapsto (\widetilde{x}, \widetilde{y}) = (\, c + \epsilon y - x^2 \, , \, c + \epsilon x - y^2 \,) \, .$$

Show that the Jacobi matrix of the mapping, defined as

$$A = \begin{pmatrix} \frac{\partial \widetilde{x}}{\partial x} & \frac{\partial \widetilde{x}}{\partial y} \\ \frac{\partial \widetilde{y}}{\partial x} & \frac{\partial \widetilde{y}}{\partial y} \end{pmatrix} ,$$

has the eigenvalues $\lambda = -x - y \pm \sqrt{(x-y)^2 + \epsilon^2}$.
(This is an example of the general theorem that the eigenvalues of a symmetric real matrix are real.)

d) Since the coupling is symmetric, so that each of the two systems influences the other system equally much, it is possible that they move exactly in step, with $x = y$. This motion, which is synchronous and in phase, is described by the one dimensional mapping (1) studied under point a) above, having two fixed points if $c > -(1-\epsilon)^2/4$.
Show that the condition we found under point a) above, that

$$c < 1 - \frac{(1-\epsilon)^2}{4} = \frac{3 + 2\epsilon - \epsilon^2}{4} \, ,$$

is necessary for the existence in the two dimensional system of a stable fixed point such that the two subsystems move synchronously and in phase.

Although this condition is necessary, it is not sufficient. Why not?
Show that another necessary condition is that

$$c < \frac{3(1-\epsilon)^2}{4} = \frac{3 - 6\epsilon + 3\epsilon^2}{4} \ .$$

This is a stronger inequality if $\epsilon > 0$ (and $|\epsilon| \ll 1$, as we assume all the time).

e) Show that there exists an "off-diagonal" two-cycle of the form

$$(u, v) \mapsto (v, u) \mapsto (u, v) \mapsto \dots \ ,$$

in which the two systems move synchronously, but in opposite phases.
Proof of the stability or instability of the two-cycle is not asked for here.
In the limiting case $u = v$ this two-cycle coincides with one of the diagonal fixed points.
For which values of the control parameters c and ϵ does this happen?
What kind of bifurcation is this? Is it subcritical or supercritical if $\epsilon > 0$?

Comment:
Of course there is also the usual "diagonal" two-cycle, of the form

$$(u, u) \mapsto (v, v) \mapsto (u, u) \mapsto \dots \ ,$$

arising from a fixed point in a flip bifurcation at $c = 1 - ((1-\epsilon)^2/4)$. This two-cycle is unstable immediately after it is born if the fixed point from which it is born is unstable.
In addition, there exist a pair of off-diagonal fixed points, of the form

$$(u, v) \mapsto (u, v) \mapsto (u, v) \mapsto \dots \qquad \text{and} \qquad (v, u) \mapsto (v, u) \mapsto (v, u) \mapsto \dots \ ,$$

but they are always unstable.
The dynamics of a system of two coupled one dimensional mappings may be surprisingly varied, even when the coupling is weak. It is, e.g., fully possible that the two two-cycles above, the diagonal and off-diagonal ones, are both stable for the same values of the control parameters, with fractal limits between the basins of attraction. Period doubling sequences and Hopf bifurcations may occur, as well as quasiperiodic motion, and *hyperchaos*, where two Lyapunov exponents are positive.

Problem 3:

a) The Korteweg–de Vries-equation (KdV equation), written in the form

$$u_t - 6uu_x + u_{xxx} = 0 \ ,$$

has the soliton solutions

$$u(x,t) = -\frac{c}{2\cosh^2\left[\frac{\sqrt{c}}{2}(x - x_0 - ct)\right]} \ . \tag{2}$$

(Notation: $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $u_{xxx} = \partial^3 u/\partial x^3$.)
Sketch briefly how these solutions may be derived.
What is the meaning of the constants c and x_0?

b) Does the KdV equation possess two-soliton solutions in which the two solitons move with exactly the same velocity, so that one follows after the other one at a constant distance?
(Hint: try e.g. the same method used for finding the one-soliton solutions of equation (2).)

c) One way to check your answer to the previous question is to study the known explicit two-soliton solutions of the equation, which are of the form

$$u(x,t) = -2\,\frac{c_1 f_1 + c_2 f_2 + C f_1 f_2[2(\sqrt{c_1}+\sqrt{c_1})^2 + c_1 f_2 + c_2 f_1]}{(1+f_1+f_2+C f_1 f_2)^2}\ ,$$

where

$$f_i = \mathrm{e}^{-\sqrt{c_i}(x-x_i^0-c_i t)}\ ,$$

and

$$C = \left(\frac{\sqrt{c_1}-\sqrt{c_2}}{\sqrt{c_1}+\sqrt{c_2}}\right)^2\ .$$

The positive constants c_1 and c_2 are the velocities of the two solitons.
What does the two-soliton solution look like in the limiting case $c_1 = c_2$?

d) Show that for a given eigenvalue λ the Schrödinger equation

$$-\psi_{xx} + u\psi = \lambda\psi\ ,$$

where $u = u(x)$ is a potential going to zero for $x \to \pm\infty$, has never more than one solution for the wave function $\psi = \psi(x)$ (up to a normalization constant), when ψ is required to be quadratically integrable.
(Hint: look at the asymptotic behaviour of $\psi(x)$ in the limits $x \to \pm\infty$.)
What is the connection between this result and the question of whether the KdV equation has two-soliton solutions in which the two solitons move with the same velocity?