THE NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF PHYSICS

Contact person:
Name: Jan Myrheim
Telephone: 93653

Examination, course TFY 4305 Nonlinear dynamics
Friday December 3, 2004
Time: 09.00–13.00

Grades made public: Friday December 24, 2004

Allowed to use: (Alternative C): Allowed pocket calculator.
Rottmann, *Mathematische Formelsammlung.*
Barnett and Cronin, *Mathematical Formulae.*
Øgrim og Lian, *Størrelser og enheter i fysikk og teknikk.*

Problem 1:

Let $u = u(x,t)$ be a complex function of one space coordinate x and one time coordinate t. The equation

$$\mathrm{i}u_t + u_{xx} + |u|^2\, u = 0 \ ,$$

where $u_t = \partial u/\partial t$ and $u_{xx} = \partial^2 u/\partial x^2$, is called the cubic Schrödinger equation.
Show that it has solutions of the form

$$u(x,t) = \mathrm{e}^{\mathrm{i}(ax-bt)}\, \varphi(x-ct) \ ,$$

where a, b and c are constants, and where φ is a real function of one variable $\xi = x - ct$ satisfying the equation

$$\frac{1}{2}\,(\varphi')^2 + \frac{b-a^2}{2}\,\varphi^2 + \frac{1}{4}\,\varphi^4 = E \ .$$

Here $\varphi' = \varphi_\xi = \mathrm{d}\varphi/\mathrm{d}\xi$, and E is an integration constant.
Is it possible to choose the constants a, b and c to be independent of each other?
Assume that $a^2 > b$. What value of E gives then a solution $u(x,t)$ which is a solitary wave?

Problem 2:

The equation of motion

$$\ddot{x} + b\dot{x} + kx + x^3 = 0 \,,$$

where b and k are constant parameters, describes the so called Duffing oscillator. The position x is a function of the time t, $\dot{x} = \mathrm{d}x/\mathrm{d}t$ is the velocity, and $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$ is the acceleration.

a) Find all the fixed points, and check whether they are stable or unstable, for all values of the parameters b and k.
What kinds of bifurcations of the fixed points take place?

b) Show that the energy

$$E = \frac{1}{2}\,\dot{x}^2 + \frac{k}{2}\,x^2 + \frac{1}{4}\,x^4$$

is conserved if $b = 0$.

c) Assume that $b > 0$. What can you then say about the sign of $\dot{E}$ (the time derivative of E)?
Are the stable fixed points globally stable?

Problem 3:

In this problem we consider the Hénon mapping

$$(x_n \,,\, y_n) \mapsto (x_{n+1} \,,\, y_{n+1}) = (1 + y_n - a x_n^2 \,,\, b x_n) \,,$$

where a and b are constant parameters. We assume here that $a > 0$ and $b > 0$.
The Jacobi matrix of the mapping is

$$A = \frac{\partial(x_{n+1} \,,\, y_{n+1})}{\partial(x_n \,,\, y_n)} = \begin{pmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{pmatrix} = \begin{pmatrix} -2ax_n & 1 \\ b & 0 \end{pmatrix} .$$

The eigenvalues of this matrix are $\lambda = -ax_n \pm \sqrt{a^2 x_n^2 + b}$.
Elimination of the y coordinates gives the following recursion formula for the x coordinates alone:

$$x_{n+1} = 1 + b x_{n-1} - a x_n^2 \,.$$

a) Show that the Hénon mapping is invertible (in other words, that x_n and y_n are single valued functions of x_{n+1} and y_{n+1}). Find explicitly the inverse mapping.

b) For which values of the parameters a and b is the Hénon mapping area preserving? Explain your reasoning.

c) Show that when $a > 0$ and $b > 0$ the mapping has precisely two fixed points, which we may call (x_+, y_+) and (x_-, y_-), and show that the x coordinates of the fixed points are

$$x_\pm = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2a} .$$

Obviously, $x_+ > 0$ and $x_- < 0$ as long as $a > 0$.

d) Show that the fixed point (x_-, y_-) is always unstable (remember that we assume all the time that $a > 0$ and $b > 0$).
Show that the second fixed point (x_+, y_+) is stable when $0 < b < 1$ and $4a < 3(b-1)^2$, and unstable when $4a > 3(b-1)^2$.
If we, for example, hold b constant, $0 < b < 1$, and increase a, so that the fixed point (x_+, y_+) goes from being stable to becoming unstable at $4a = 3(b-1)^2$, what kind of bifurcation is this?

e) For which parameter values does there exist a two-cycle, of the form
$(x_1, y_1) \mapsto (x_2, y_2) \mapsto (x_1, y_1) \mapsto (x_2, y_2) \mapsto \ldots$?
If we again hold b constant, $0 < b < 1$, and vary a, so that this two-cycle is born at one particular critical value of a, what kind of bifurcation is this?
(If you do not know the answer, guessing is allowed!)

f) We now fix the parameter values $a = 1.4$ and $b = 0.3$. Figure 1, see the last page, shows a quadrilateral ABCD, with the corners $A = (-1.06\,, -0.5)$, $B = (1.245\,, -0.14)$, $C = (1.32\,, 0.133)$, and $D = (-1.33\,, 0.42)$.
This quadrilateral is a trapping region for the Hénon mapping with these parameter values. This claim is proved by the figure, since the quadrilateral ABCD is mapped onto a region bordered by parabolic curves, which is drawn in the figure, and which lies wholly within the quadrilateral.
What is the area of this region, given that the quadrilateral region has an area of 1.48?
Show, by drawing a figure, which points A', B', C' and D' the four corners A, B, C and D are mapped into.
Explain, by means of a similar figure, and maybe several figures, how the mapping creates a strange attractor having a dimension between one and two.
The figure shows the two fixed points, marked by plus signs and the letter "f". You may also mark in your figure (approximately) where the two points belonging to the two-cycle are located.

Figure:

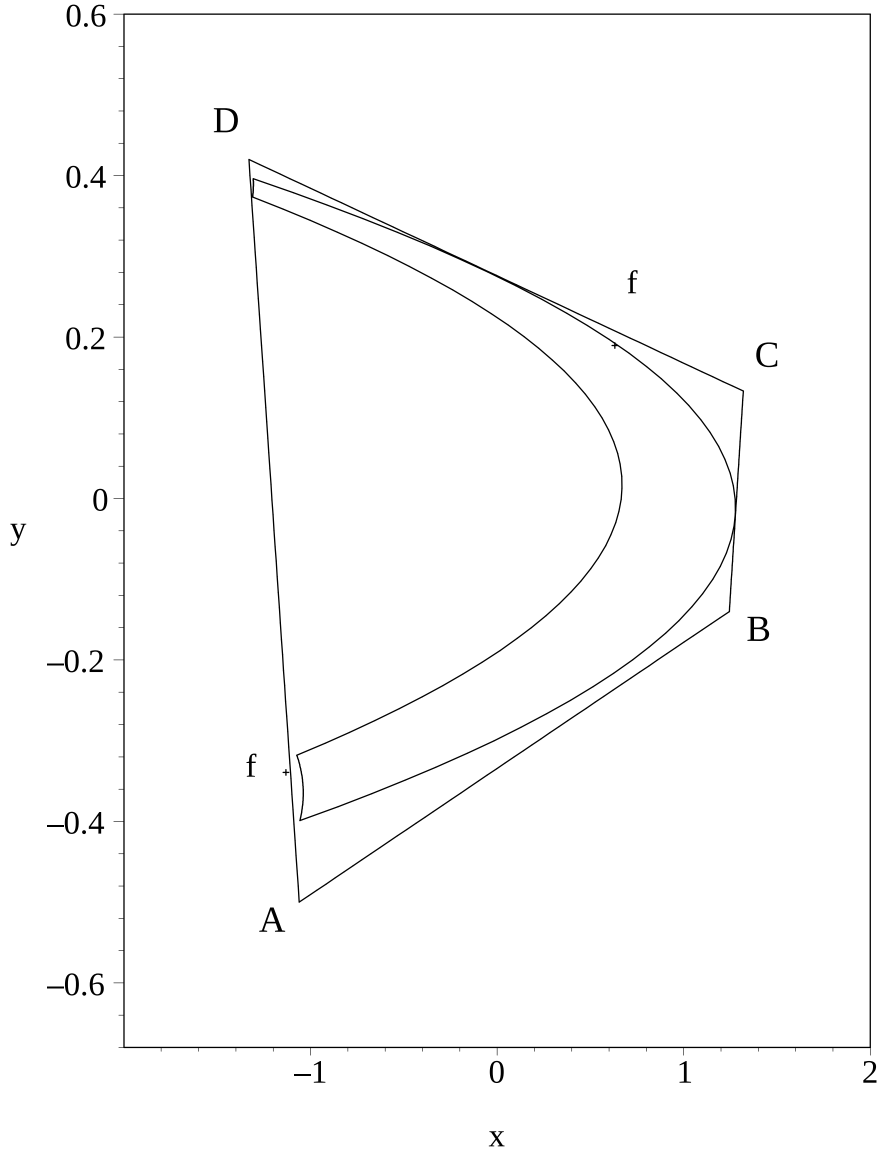

Figure 1: The quadrilateral ABCD is a trapping region of the Hénon mapping with the parameter values $a = 1.4$, $b = 0.3$. Two fixed points are marked by the symbol "+" and the letter "f".