THE NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF PHYSICS

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Examination, course TFY 4305 Nonlinear dynamics
Wednesday November 30, 2005
Time: 15.00–19.00

Grades made public: Wednesday December 21, 2005

Allowed to use: (Alternative C): Allowed pocket calculator.
Rottmann, *Mathematische Formelsammlung.*
Barnett and Cronin, *Mathematical Formulae.*
Øgrim og Lian, *Størrelser og enheter i fysikk og teknikk.*

Problem 1:

Given the eigenvalue equation $L\psi = \lambda\psi$, where L is the following Schrödinger operator,

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x) \ ,$$

while λ is an eigenvalue, and $\psi = \psi(x)$ is a (real or complex) eigenfunction.
The function $u = u(x)$ is real, and is interpreted as a potential when the equation $L\psi = \lambda\psi$ is interpreted as the time independent (stationary) Schrödinger equation.
Assume now that u depends also on time t, that is, $u = u(x,t)$, and satisfies the equation

$$u_t + cu_x = 0 \ ,$$

where c is constant, and $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$.
Show that it is then possible to let also ψ be time dependent, $\psi = \psi(x,t)$, in such a way that the eigenvalue equation $L\psi = \lambda\psi$ holds with a constant (time independent) eigenvalue λ.
You may show this in two different ways:
1) By finding the general solution of the equation $u_t + cu_x = 0$.
2) By finding an operator A such that L and A form a Lax pair of operators, with

$$[L, A] = cu_x \ ,$$

whereas the time dependence of the eigenfunction ψ is given by the equation $\psi_t = A\psi$.

Problem 2:

The basic process going on in a nuclear reactor, or a nuclear bomb of the fission type, is that a neutron is absorbed in a heavy atomic nucleus, for example uranium 235 or plutonium 239, inducing the nucleus to fission into two or more lighter nuclei. Binding energy is liberated in the process, as well as a number of neutrons that may induce the fissioning of other nuclei, and so on in a chain reaction. Such a chain reaction is a statistical process, because the number of neutrons liberated in one fissioning event varies statistically.
In order for the reactor, or bomb, to function, it is necessary that the chain reaction does not stop by itself after only a few steps. In this problem we will examine a method for computing the probability of one neutron starting a chain reaction that does not stop.
Let p_n, with $n = 0, 1, 2, \ldots$, be the probability for one neutron to cause a nuclear fission event in which n new neutrons are produced. Thus, for example, p_0 is the probability that the one neutron either does not cause a fissioning, or else causes a fissioning with no new neutrons produced.
A necessary condition for the occurrence of a chain reaction is obviously that at least once in a while there is more than one neutron liberated when a nucleus fissions. We therefore assume that $p_n > 0$ for at least one $n > 1$.
Define the *probability generating function*

$$F(x) = p_0 + p_1 x + \cdots + p_n x^n + \cdots = \sum_{n=0}^{\infty} p_n x^n \ ,$$

where x is a variable the only purpose of which is to serve as an argument of the function F. Because probabilities are always non-negative, $F(x)$ and all its derivatives are non-negative for $x \geq 0$. Since we assume that $p_n > 0$ for at least one $n > 1$, the second derivative satisfies the stronger inequality

$$F''(x) > 0 \quad \text{for all} \quad x > 0 \ . \tag{1}$$

The total probability equals 1, and this means that $x = 1$ is a fixed point of F,

$$F(1) = \sum_{n=0}^{\infty} p_n = 1 \ .$$

The probability generating function contains all the information about the probability distribution. For example, the expectation value of the number of neutrons that are first generation descendants from the single neutron, may be computed by differentiation of F,

$$\langle n \rangle = \sum_{n=0}^{\infty} p_n \, n = F'(1) \ .$$

We define the first neutron as generation 0, and we say that a neutron belongs to generation $g = 1, 2, 3, \ldots$ if it is produced in a nuclear fission caused by a neutron of generation $g - 1$. Let $p(n, g)$ be the probability for generation g to consist of n neutrons. The probability generating function of generation g is

$$F_g(x) = \sum_{n=0}^{\infty} p(n, g) \, x^n \ .$$

The total probability in each generation equals 1, this means that

$$F_g(1) = \sum_{n=0}^{\infty} p(n,g) = 1 \; .$$

In our notation, we have that $F_0(x) = x$ and $F_1(x) = F(x)$.
If we for example know that generation g consists of two neutrons, then we have per definition that $F_g(x) = x^2$, and it is easy to show that $F_{g+1}(x) = (F(x))^2$ (you are not asked to prove this result here).
More generally, if we know that generation g consists of k neutrons, then we have per definition that $F_g(x) = x^k$, and it is almost as easy to prove (by induction) that we have $F_{g+1}(x) = (F(x))^k$.
The relation

$$F_{g+1}(x) = F_g(F(x))$$

holds quite generally, also when we only know a probability distribution for the number of neutrons in generation g. Hence, the probability generating function F_g for generation g equals the g times iterated probability generating function F,

$$F_g(x) = F^g(x) = F(F(\ldots F(x)\ldots)) \; .$$

Even though this process is statistical, it may be treated by means of iterations of a function, and therefore we may use methods from deterministic nonlinear dynamics.

a) The function $F(x)$ has, as stated above, always the fixed point $x = x_1 = 1$.

Show that if $F'(1) = 1$, then x_1 is the only fixed point for $x \geq 0$.

Remember the inequality given in equation (1).

The fixed point x_1 is marginally stable when $F'(1) = 1$.
Decide, for example by drawing a figure, whether it is stable or unstable.

What is the physical interpretation of the condition $F'(1) = 1$?

b) Show that if $F'(1) \neq 1$, then $F(x)$ has exactly one fixed point $x = x_2 \geq 0$ in addition to x_1.

Show that $x_2 > 1$ if $F'(1) < 1$, and that $0 \leq x_2 < 1$ if $F'(1) > 1$. Draw a figure, if you find that helpful.

Is the fixed point x_2 stable or unstable?

c) The expectation value of the number of neutrons in generation g is

$$\langle n \rangle_g = \sum_{n=0}^{\infty} p(n,g)\, n = F_g'(1) \; .$$

Show that the number of neutrons grows (or decreases) exponentially with time, in the sense that the expectation value $\langle n \rangle_g$ grows (or decreases) exponentially as a function of the generation number g. More precisely, show that

$$\langle n \rangle_g = (\langle n \rangle_1)^g \; .$$

d) Even though the expected number of neutrons grows exponentially, it may very well happen that the probability is larger than zero for the whole chain reaction to stop by itself.

The probability for the chain reaction to stop before generation g, is

$$p(0, g) = F_g(0) \ .$$

The probability for it to stop after a finite number of generations, is

$$p(0, \infty) = \lim_{g \to \infty} p(0, g) \ .$$

Thus, the probability for the chain reaction not stopping, is $1 - p(0, \infty)$. In order for a nuclear reactor or a nuclear bomb to function, it has to be constructed in such a way that $p(0, \infty) < 1$.

Find a necessary and sufficient condition that the probability distribution p_n has to fulfill in order that $p(0, \infty) < 1$.

As a numerical example, assume that $p_0 = 0.18$, $p_1 = 0.58$, $p_2 = 0.24$, and $p_n = 0$ for $n > 2$.

How large is then the expectation value $\langle n \rangle_g$ for $g = 1$? For $g = 10$? $g = 100$? $g = 1000$?

And how large is the stopping probability $p(0, \infty)$?

Comment: The limit $g \to \infty$ is of course an unrealistic mathematical abstraction. The important question in practice is whether the chain reaction continues long enough that the number of atomic nuclei involved is a substantial part, say a few per cent, of the total number present. The mathematical limit $p(0, \infty)$ is a good estimate of the probability for that to happen.

Problem 3:

The Lorenz equations

$$\begin{aligned}\dot{x} &= \sigma(y - x)\ ,\\ \dot{y} &= (r - z)x - y\ ,\\ \dot{z} &= xy - bz\ ,\end{aligned}$$

contain the three constant parameters $\sigma > 0$, $b > 0$ and $r > 0$.

a) Show that the Lorenz equations are invariant under the transformation $(x, y, z) \mapsto (-x, -y, z)$, which is a rotation of 180° about the z axis.

What is the significance of this symmetry concerning, for example, fixed points, attractors and bifurcations?

Figure 1 (see the last page) shows, as examples, two dimensional projections of attractors in the Lorenz system with the "standardized" parameter values $\sigma = 10$, $b = 8/3$, and with the four different values $r = 240$, $r = 220$, $r = 216$ and $r = 214$. The figure shows that there exist two distinct attractors in each of these four examples.

b) Does the Lorenz system have time reversal symmetry? Give reasons for your answer.

Why can the Lorenz equations not have any quasiperiodic solution?

c) Show that the origin is a stable fixed point for $r < 1$, and an unstable fixed point, more precisely a three dimensional saddle point, for $r > 1$.

Which other fixed points exist?

What type of bifurcation takes place when the origin becomes unstable at $r = 1$? Here much work is required for a strict justification of the answer, hence a qualified guess is accepted.

A comment (for confusion or enlightenment): Let $\sigma = 10$ and $b = 8/3$. At a certain critical parameter value $r = r_c \approx 313$ the same type of bifurcation takes place, but with *decreasing* r instead of with *increasing* r, and with *limit cycles* instead of with *fixed points*. For $r > r_c$ there exists a globally stable limit cycle which is symmetric about $(x, y) = (0, 0)$. At $r = r_c$ it becomes unstable, at the same time as two new limit cycles are created, both existing and stable for $r < r_c$. Each of the two new limit cycles, taken separately, is unsymmetrical about $(x, y) = (0, 0)$. For $r < r_c$ the symmetric limit cycle still exists, but it is not seen in Figure 1 because it is unstable.

d) Figure 1 (last page) shows the attractors in the Lorenz system with $\sigma = 10$, $b = 8/3$, and with $r = 240$, $r = 220$, $r = 216$ and $r = 214$.

Which of the attractors in the figure are limit cycles, and which are chaotic?

What type of bifurcation is this an example of?

Figure:

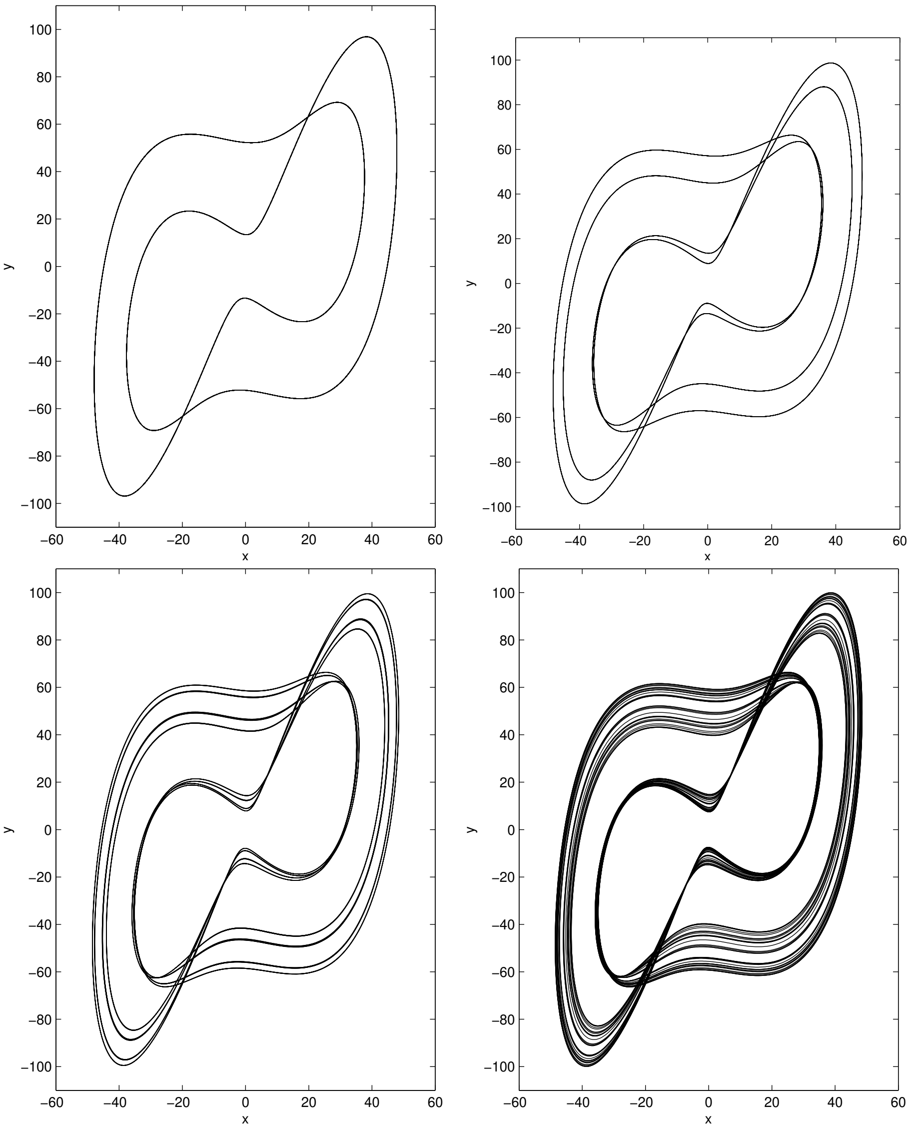

Figure 1: Attractors in the Lorenz system, projected on the xy plane, for four values of the parameter r. Upper left: $r = 240$; upper right: $r = 220$; lower left: $r = 216$; and lower right: $r = 214$. The parameter values $\sigma = 10$ and $b = 8/3$ are common for all the four plots. Each plot shows two distinct attractors, which may be reached from different starting points.