Exam 18.12.02 in SIF4088 Nonlinear dynamics
Solutions

Problem 1

a) A permanent wave is of the form $u(x - ct)$, with constant velocity c; it propagates with a permanent shape.

A solitary wave is a *localized* permanent wave, in the sense that all essential changes of its shape occur over a finite interval of x.

Solitons are solitary waves that survive collisions with each other.

Different type solitons: (i) Pulseformed solitons, as for the KdV equation, (ii) Envelope solitons, as for the cubic Schrödinger equation, (iii) Kink solitons, whose gradient has pulse form, as for the sine-Gordon equation.

(Sketch omitted here)

b) and **c)**: As lecture notes.

d) KdV applies to one-dimensional flow on shallow water, when the height of the surface wave is much smaller, and the extension of the wave is much longer, than the constant water depth.

Problem 2

a) Fixed points for

$$1 - \mu x^2 = x,$$

with solutions

$$x^* = (-1 \pm \sqrt{1 + 4\mu})/(2\mu).$$

A fixed point x^* of an iteration $x_{n+1} = F(x_n)$ is stable when $|F'(x^*)| < 1$. In our case

$$F'(x^*) = -2\mu x^* = -\left(-1 \pm \sqrt{1 + 4\mu}\right).$$

Clearly the negative fixed point is always unstable. The positive fixed point

$$x^* = (-1 + \sqrt{1 + 4\mu})/(2\mu) \tag{1}$$

has the limiting value $F'(x^*) = -1$ for $\sqrt{1 + 4\mu} = 2$, i.e.

$$\mu = \mu_1 = \underline{\underline{\tfrac{3}{4}}}.$$

For $\mu > \mu_1$ this fixed point is unstable, for $0 \leq \mu < \mu_1$ the fixed point is stable.

b) The period 2 values satisfy

$$
\begin{aligned}
x_- &= 1-\mu x_+^2 \qquad (2)\\
x_+ &= 1-\mu x_-^2 \qquad (3)
\end{aligned}
$$

By subtraction we obtain

$$x_- - x_+ = \mu(x_-^2 - x_+^2) = \mu(x_- - x_+)(x_- + x_+).$$

Assuming $x_- - x_+ \neq 0$ this gives

$$1 = \mu(x_- + x_+),$$

as should be shown. Inserting $x_- = 1/\mu - x_+$ into (2) we obtain

$$\mu^{-1} - x_+ = 1 - \mu x_+^2,$$

and for x_- we obtain the same equation. Solving, we get

$$x_\pm = \frac{1 \pm \sqrt{4\mu - 3}}{2\mu},$$

as should be shown.

c) The stability of period two requires $|dF(F(x))/dx| < 1$ on the attractor. Since

$$dF(F(x_+))/dx = F'(x_-)F'(x_+) = 4\mu^2 x_- x_+ = 4 - 4\mu,$$

the maximum value of μ corresponds to this derivative being -1, i.e.

$$\mu = \mu_2 = \underline{\underline{\tfrac{5}{4}}}.$$

Superstability is when the derivative of $F(F(x))$ vanishes, which occurs for

$$\mu = \underline{\underline{1}}.$$

One could also have argued that the superstable orbit had to visit $x_- = 0$ where $F' = 0$, and therefore also $x_+ = F(0) = 1$. Thus $\mu = x_- + x_+ = 1$.

d) The Lyapunov exponent measures, on an exponential scale, the rate with which two neighbouring initial values either separate (for $\lambda > 0$) or come closer (for $\lambda < 0$) under iteration.

For our iteration the period-two attractor is stable in the interval $\mu_1 < \mu < \mu_2$, and consequently $\lambda < 0$. The end points of the interval correspond to marginal stability so that $\lambda = 0$ here. At the superstable value the stability is infinitely high, so that $\lambda = -\infty$ for $\mu = 1$.

(Sketch omitted here.)

Problem 3

a) Our system is two-dimensional, and in two dimensions (for an autonomous system) the only attractors are fixed points and limit cycles (Poincaré- Bendixson). Thus no chaotic attractor can exist.

b) When $b_A = b_B = 0$ the species develop independently. Since $dn_A/dt < 0$ for $n_A > r_A/a_A$, and $dn_A/dt > 0$ for $n_A < r_A/a_A$, the population will approach the fixed point r_A/a_A from above or from below depending on the initial value $n_A(0)$, as long as this is positive. Similarly with species B. Thus

$$n_A(\infty) = r_A/a_A, \qquad \text{and} \qquad n_B(\infty) = r_B/a_B.$$

c) By introducing $n_A = x\, r/a_A$ and $n_B = y\, r/a_B$ and $rt = \tau$ we have

$$\begin{aligned} \frac{dx}{d\tau} &= x(1 - x - yb_A/a_B) \\ \frac{dy}{d\tau} &= y(1 - y - xb_B/a_A) \end{aligned} \tag{4}$$

Hence

$$\alpha = b_A/a_B \qquad \text{and} \qquad \beta = b_B/a_A.$$

d) The fixed points correspond to the right-hand sides of (4) being zero:

$$x(1 - x - \alpha y) = 0 \tag{5}$$

$$y(1 - y - \beta x) = 0 \tag{6}$$

There are four solutions, four fixed points. Three are on the axes:

$$F_1 = (0,0); F_2 = (0,1); F_3 = (1,0).$$

The fourth, $F_0 = (x^0, y^0)$ corresponds to

$$1 - x^0 - \alpha y^0 = 0, \qquad 1 - y^0 - \beta x^0 = 0.$$

The solution of these two linear equations with two unknowns is

$$x^0 = \frac{1-\alpha}{1-\alpha\beta}, \qquad y^0 = \frac{1-\beta}{1-\alpha\beta}.$$

The fixed point F_0 is in the physical relevant region when both x^0 and y^0 are nonnegative. This is the case when $\alpha \geq 1,\ \beta \geq 1$ or $\alpha \leq 1,\ \beta \leq 1$.

e) We linearize in each case the equations near the fixed point. We could do it case by case, we see for instance that near the fixed point at the origin linearization gives

$$dx/d\tau = x, \qquad dy/d\tau = y,$$

with solution

$$x(\tau) = x(0)e^{\tau}, \qquad y(\tau) = y(0)e^{\tau} = \frac{y(0)}{x(0)}\,x.$$

The phase point is repelled along straight lines, the origin is consequently an unstable (repelling) node.

Let us alternatively look at the eigenvalues of the Jacobian matrix. The Jacobian for the dynamical system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ is in our case

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 - 2x - \alpha y & -\alpha x \\ -\beta y & 1 - 2y - \beta x \end{pmatrix}. \tag{7}$$

We must insert the fixed point coordinates, for the four fixed points:

$$\boxed{F_1 = (0, 0)}$$

Here

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{with eigenvalues } \lambda_1 = 1, \lambda_2 = 1.$$

Two real positive eigenvalues correspond to an unstable node, as already noted.

$$\boxed{F_2 = (0, 1)}$$

At $F_2 = (0, 1)$ we have

$$J = \begin{pmatrix} 1 - \alpha & 0 \\ -\beta & -1 \end{pmatrix}, \qquad \text{with eigenvalues } \lambda_1 = 1 - \alpha, \lambda_2 = -1.$$

Thus, for $\alpha < 1$ there are two real eigenvalues of opposite sign, which implies that F_2 is a saddle point. For $\alpha > 1$, however, there are two real negative eigenvalues, which implies that F_2 is a stable (attracting) node in this case.

$$\boxed{F_3 = (1, 0)}$$

At $F_3 = (1, 0)$ we have

$$J = \begin{pmatrix} -1 & -\alpha \\ 0 & 1 - \beta \end{pmatrix}, \qquad \text{with eigenvalues } \lambda_1 = -1, \lambda_2 = 1 - \beta.$$

Thus, for $\beta < 1$ there are two real eigenvalues of opposite sign, which implies that F_3 is a saddle point. For $\beta > 1$, however, there are two real negative eigenvalues, which implies that F_3 is a stable (attracting) node in this case.

$$\boxed{F_0 = (\tfrac{1-\alpha}{1-\alpha\beta}, \tfrac{1-\beta}{1-\alpha\beta})}$$

Here we have

$$J = \begin{pmatrix} \frac{\alpha-1}{1-\alpha\beta} & -\frac{\alpha(1-\alpha)}{1-\alpha\beta} \\ -\frac{\beta(1-\beta)}{1-\alpha\beta} & \frac{\beta-1}{1-\alpha\beta} \end{pmatrix}.$$

The eigenvalues are

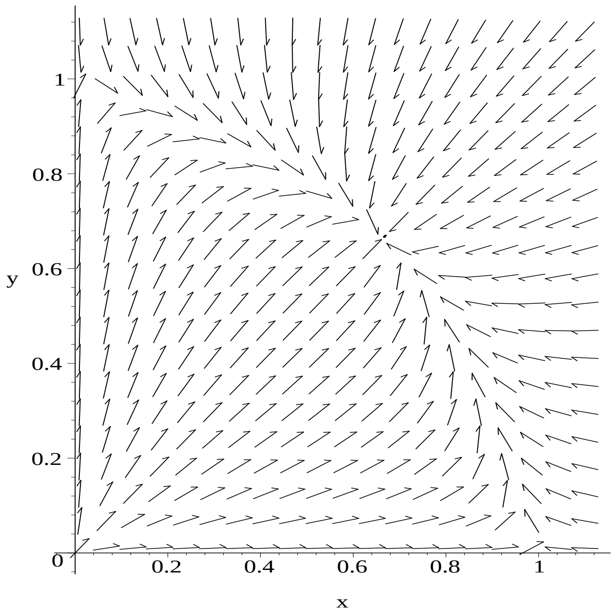
$$\lambda = \left(\frac{1}{2}\alpha + \frac{1}{2}\beta - 1 \pm \sqrt{\frac{1}{4}(\alpha-\beta)^2 + \alpha\beta(1-\alpha)(1-\beta)}\right) / (1-\alpha\beta).$$

We found above that F_0 is in the physical interesting region when both control parameters are either less than 1 or both greater than 1.

When both α and β are *less* than unity, the two eigenvalues are both real and negative, so that F_0 is then a stable node.

When both α and β are *greater* than unity, the eigenvalues are real and of opposite sign, so that F_0 is a saddle point in this case.

f) For $\alpha = \beta = \frac{1}{2}$ the fixed point $F_0 = (\frac{2}{3}, \frac{2}{3})$ is a stable (attracting) node.

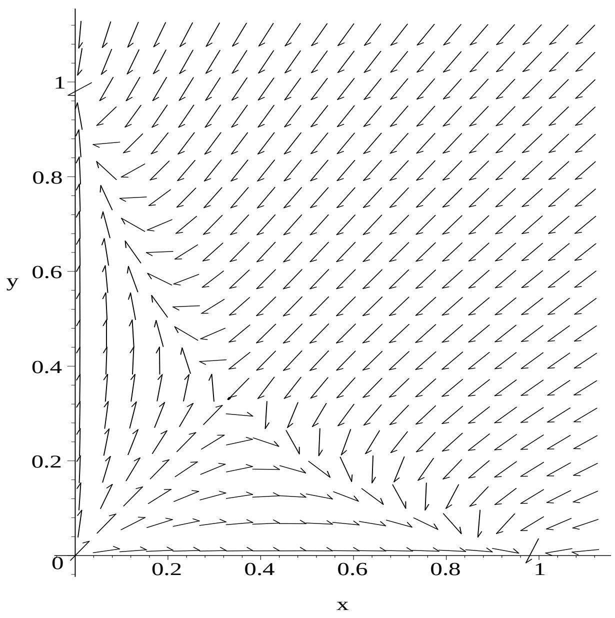


If both initial values are positive, the system ends up at the stable node F_0, i.e. $x(\infty) = y(\infty) = \frac{2}{3}$. There is *coexistence* between the two species.

g) For $\alpha = \beta = 2$ the fixed point $F_0 = (\frac{1}{3}, \frac{1}{3})$ is a saddle point. The diagonal is the attracting direction, which can be seen directly from the equations:

$$\frac{dy}{d\tau} = \frac{dx}{d\tau} = x(1-3x) \text{ for } x = y,$$

and $x = \frac{1}{3}$ is clearly attracting. Near (but not on) the diagonal the phase point is repelled, and the only final possibility is one of the fixed points F_2 or F_3, depending on which side of the diagonal the initial point is located.



The conclusion is as follows: If the initial densities should be *exactly* equal and nonzero the system ends up at the saddle point F_0. Otherwise, and more realistically, the system ends up with either $x(\infty) = 1,\ y(\infty) = 0$ or with $x(\infty) = 0,\ y(\infty) = 1$. In each case *the minority species becomes extinct.*