TFY 4305 Nonlinear dynamics, autumn 2005.
Solutions to exercises

Strogatz, exercise 2.2.13

The equation of motion is

$$m\dot{v} = mg - kv^2 \,,$$

where v is the velocity of the skydiver, m is the mass, g is the acceleration of gravity, and k is called the drag constant. As always, $\dot{v} = \mathrm{d}v/\mathrm{d}t$ is the time derivative.
We assume that $v \geq 0$, defining downwards as the positive direction. If $v < 0$, we would have $m\dot{v} = mg + kv^2$.

a) To solve the equation, we separate the variables v and t, writing

$$\frac{m\,\mathrm{d}v}{mg - kv^2} = \mathrm{d}t \,.$$

Introducing an integration constant t_0, we get that

$$\begin{aligned} t - t_0 &= \int \mathrm{d}t = \int \frac{m\,\mathrm{d}v}{mg - kv^2} = \int \frac{m\,\mathrm{d}v}{2\sqrt{mg}} \left(\frac{1}{\sqrt{mg} + \sqrt{k}\,v} + \frac{1}{\sqrt{mg} - \sqrt{k}\,v} \right) \\ &= \frac{m}{2\sqrt{mgk}} \left(\ln\left|\sqrt{mg} + \sqrt{k}\,v\right| - \ln\left|\sqrt{mg} - \sqrt{k}\,v\right| \right) \\ &= \frac{\sqrt{m}}{2\sqrt{gk}} \ln\left| \frac{\sqrt{mg} + \sqrt{k}\,v}{\sqrt{mg} - \sqrt{k}\,v} \right| . \end{aligned}$$

Let us introduce the constants

$$V = \sqrt{\frac{mg}{k}} \,, \qquad T = \frac{1}{\lambda} = \frac{V}{g} = \sqrt{\frac{m}{gk}} \,.$$

We see that V has the dimension of velocity, and T has the dimension of time. We choose $t_0 = 0$, that is, we start our clocks at the time t_0. Then we have the equation

$$t = \frac{T}{2} \ln\left|\frac{V+v}{V-v}\right| = \frac{1}{2\lambda} \ln\left|\frac{V+v}{V-v}\right| ,$$

or, assuming $0 \leq v < V$,

$$\mathrm{e}^{2\lambda t} = \frac{V+v}{V-v} \,.$$

Which we solve for v, getting

$$v = v(t) = V\,\frac{\mathrm{e}^{2\lambda t} - 1}{\mathrm{e}^{2\lambda t} + 1} = V\,\frac{\mathrm{e}^{\lambda t} - \mathrm{e}^{-\lambda t}}{\mathrm{e}^{\lambda t} + \mathrm{e}^{-\lambda t}} = V\,\tanh(\lambda t) = V\,\tanh\!\left(\frac{gt}{V}\right) .$$

This solution for $v(t)$ satisfies the initial condition $v(0) = 0$.

b) Since $v(t) \to V$ as $t \to \infty$, the physical meaning of $V = \sqrt{mg/k}$ is that it is the terminal velocity.

c) The equation is $\dot{v} = f(v)$ with

$$f(v) = g - \frac{kv^2}{m} \ .$$

The fixed point equation, $f(v) = 0$, has the unique solution $v = \sqrt{mg/k} = V$ (remember our restriction that $v \geq 0$).
This fixed point is stable, since the derivative $f'(v) = -2kv/m$ is negative at $v = V$. Hence, v will converge to the fixed point V, in other words, V is the terminal velocity.

d) The average velocity is

$$V_{\mathrm{avg}} = \frac{31400 \text{ ft} - 2100 \text{ ft}}{116 \text{ s}} = 252.6 \text{ ft/s} = 252.6 \times 0.3048 \text{ m/s} = 77.0 \text{ m/s} \ .$$

Conversion: $1 \text{ ft} = 12 \text{ in} = 12 \times 2.54 \text{ cm} = 30.48 \text{ cm}$.

e) The distance fallen is

$$s(t) = \int_0^t \mathrm{d}t' \ v(t') = V \int_0^t \mathrm{d}t' \ \tanh\!\left(\frac{gt'}{V}\right) = \frac{V^2}{g} \ln\cosh\!\left(\frac{gt}{V}\right) .$$

If $x \gg 1$, then clearly

$$\ln\cosh x = \ln\!\left(\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}\right) \approx \ln\!\left(\frac{\mathrm{e}^x}{2}\right) = x - \ln 2 \ .$$

With this (very good) approximation, we have that

$$s = \frac{V^2}{g}\left(\frac{gt}{V} - \ln 2\right) = Vt - \frac{V^2 \ln 2}{g} \ .$$

Or rewritten,

$$V^2 - \frac{gt}{\ln 2} V + \frac{gs}{\ln 2} = 0 \ .$$

The two solutions of this equation are

$$V = \frac{gt}{2\ln 2} \pm \sqrt{\left(\frac{gt}{2\ln 2}\right)^2 - \frac{gs}{\ln 2}} \ .$$

The positive sign in front of the square root gives a value for V which depends on t, more specifically, it increases linearly with t at large t, and this does not make sense. The negative sign gives a value for V going to a constant at large t, this is the one we want. However, if we compute

$$V = \frac{gt}{2\ln 2} - \sqrt{\left(\frac{gt}{2\ln 2}\right)^2 - \frac{gs}{\ln 2}}$$

as it stands, at large t, we get the answer as a small difference between two large quantities, and from a numerical point of view this is not the best way to compute. A better way is to write instead

$$
\begin{aligned}
V &= \frac{\left(\frac{gt}{2\ln 2} - \sqrt{\left(\frac{gt}{2\ln 2}\right)^2 - \frac{gs}{\ln 2}}\right)\left(\frac{gt}{2\ln 2} + \sqrt{\left(\frac{gt}{2\ln 2}\right)^2 - \frac{gs}{\ln 2}}\right)}{\frac{gt}{2\ln 2} + \sqrt{\left(\frac{gt}{2\ln 2}\right)^2 - \frac{gs}{\ln 2}}} \\
&= \frac{\frac{gs}{\ln 2}}{\frac{gt}{2\ln 2} + \sqrt{\left(\frac{gt}{2\ln 2}\right)^2 - \frac{gs}{\ln 2}}} = \frac{2gs}{gt + \sqrt{(gt)^2 - 4gs\ln 2}} \ .
\end{aligned}
$$

In the limit of large t we see that

$$V \approx \frac{2gs}{2gt} = \frac{s}{t} = V_{\text{avg}} = 77.0 \text{ m/s} \ .$$

But remember that we made an approximation to arrive at this result. The more accurate answer for the terminal velocity is

$$V = \sqrt{\frac{mg}{k}} = 265.7 \text{ ft/s} = 81.0 \text{ m/s} \ .$$

Strogatz, exercise 2.3.1

The logistic equation,

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) ,$$

with positive constants r (growth rate for a small population) and K (carrying capacity) is separable, and may be written as

$$r\,\mathrm{d}t = \frac{K\,\mathrm{d}N}{N(K-N)} = \mathrm{d}N\left(\frac{1}{N} + \frac{1}{K-N}\right) .$$

Integrating and introducing an integration constant t_0 we get

$$r(t-t_0) = \ln|N| - \ln|K-N| = \ln\left|\frac{N}{K-N}\right| .$$

If $0 < N < K$, then we get that

$$\frac{N}{K-N} = \mathrm{e}^{r(t-t_0)} \ ,$$

hence,

$$N = \frac{K\,\mathrm{e}^{r(t-t_0)}}{\mathrm{e}^{r(t-t_0)} + 1} = \frac{K}{1 + \mathrm{e}^{-r(t-t_0)}} \ .$$

In the two limits $t_0 \to \pm\infty$ we get $N = 0$ or $N = K$, which are actually two constant solutions of the original equation.

Another possibility is that $0 < K < N$, then we get that

$$\frac{N}{N-K} = \mathrm{e}^{r(t-t_0)} \ ,$$

hence,

$$N = \frac{K\,\mathrm{e}^{r(t-t_0)}}{\mathrm{e}^{r(t-t_0)} - 1} = \frac{K}{1 - \mathrm{e}^{-r(t-t_0)}}\ .$$

For $t < t_0$ this gives $N < 0$, which is a mathematical (but unphysical) solution of the equation.

The second method suggested for solving the equation is to substitute $x = 1/N$, this gives

$$\dot{x} = -\frac{\dot{N}}{N^2} = -\frac{rN(K-N)}{KN^2} = -\frac{r(K-N)}{KN} = -rx + \frac{r}{K}\ ,$$

an inhomogeneous linear ordinary differential equation for $x(t)$. We may solve it by separation of variables. Or else we note that $x(t) = 1/K$ is one particular solution, and the general solution of the homogeneous equation $\dot{x} = -rx$ is $x(t) = \mathrm{e}^{-r(t-t_0)}$. Hence, the general solution of the inhomogeneous equation is

$$x = \frac{1}{N} = \frac{1}{K} + \mathrm{e}^{-r(t-t_0)}\ .$$

This general solution with t_0 real and finite, has $0 < N < K$, and N increasing towards K. However, at this point we have to be careful. In fact, since t_0 is arbitrary, it could be $-\infty$, which gives the constant solution

$$x = \frac{1}{N} = \frac{1}{K}\ .$$

And t_0 could even be complex, for example

$$t_0 = t_0' + \mathrm{i}\,\frac{\pi}{r}\ ,$$

with t_0' real, so that we get a decreasing solution for N,

$$x = \frac{1}{N} = \frac{1}{K} - \mathrm{e}^{-r(t-t_0')}\ .$$

Strogatz, exercise 2.3.3 and 2.4.8

The Gompertz law for tumor growth, $\dot{N} = -aN\ln(bN)$, may be rewritten like this:

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln(bN) = -a\ln(bN)\ .$$

Here a and b are positive constants. The general solution is

$$\ln(bN) = C\,\mathrm{e}^{-at}\ ,$$

where C is a constant which may be positive, negative, or zero. The limiting value as $t \to \infty$ is anyway that $\ln(bN) = 0$, hence $bN = 1$, which shows that $N = 1/b$ is the only fixed point with $N > 0$, and it is stable.

Actually, $N = 0$ is a fixed point of the original equation $\dot{N} = -aN\ln(bN)$. It is true that $\ln(bN) \to -\infty$ as $N \to 0$, but it is also true that $N\ln(bN) \to 0$ as $N \to 0$.

The "double exponential" time dependence

$$N(t) = \frac{1}{b}\,\mathrm{e}^{C\,\mathrm{e}^{-at}}$$

may seem strange. It means for $t \approx 0$ that

$$N(t) \approx \frac{1}{b}\,\mathrm{e}^{C(1-at)} = \frac{\mathrm{e}^C}{b}\,\mathrm{e}^{-Cat}\;,$$

and for $t \to \infty$ that

$$N(t) \approx \frac{1}{b}\,(1 + C\,\mathrm{e}^{-at})\;.$$

For $t \approx 0$ we have the exponential growth rate $\dot{N}/N = -Ca$, which is positive when C is negative.
In the limit $t \to \infty$ we have exponential convergence to the fixed point $N = 1/b$, with a different rate a.

The parameter a determines the time scale in the system, it is (proportional to) the cell division rate, i.e., the number of cell generations per time. More precisely: we may choose $C = \ln b$ such that $N = 1$ for $t = 0$, and we see then that the growth rate for a single cancer cell is $-Ca = -a \ln b$. Which we may also see more directly by setting $N = 1$ in the equation $\dot{N} = -aN \ln(bN)$.

If N is the number of cancer cells in the tumor, then $1/b$ is the number of cells in the tumor when it has a stable size and is neither growing nor shrinking.

Linear stability analysis for the fixed point $N = 1/b$ gives the equation $\dot{n} = \lambda n$, where

$$N = \frac{1}{b} + n\;,$$

and n is small, and where

$$\lambda = \frac{\mathrm{d}}{\mathrm{d}N}(-aN\ln(bN))\bigg|_{N=1/b} = (-a\ln(bN) - a)|_{N=1/b} = -a\;.$$

This shows that the fixed point is stable (we assumed $a > 0$).

Strogatz, exercises 2.4.2, 2.4.4 and 2.4.7

Given the equation $\dot{x} = f(x)$. The fixed points are given by the equation $f(x) = 0$.
Linear stability analysis says that a fixed point x_* is stable if $f'(x_*) < 0$ and unstable if $f'(x_*) > 0$.
In the case $f'(x_*) = 0$, linear stability analysis is insufficient (but we may look at the sign of the first of the derivatives which is nonzero).

2.4.2 $f(x) = x(1-x)(2-x) = x(x-1)(x-2)$, $f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$.
Fixed points are $x = 0$, $x = 1$ and $x = 2$. Since $f'(0) = 2$, $f'(1) = -1$ and $f'(2) = 2$, $x = 1$ is stable, $x = 0$ and $x = 2$ are unstable.

2.4.4 $f(x) = x^2(6-x)$, $f'(x) = 12x - 3x^2 = 3x(4-x)$, $f''(x) = 12 - 6x$.
Fixed points are $x = 0$ and $x = 6$. Since $f'(0) = 0$, and $f'(6) = -36$, $x = 6$ is stable, $x = 0$ needs a closer look. Since $f''(0) = 12$, we have $f(x) > 0$ for x close to 0, $x \neq 0$. Hence the fixed point $x = 0$ is half stable: stable from the left ($x < 0$), unstable from the right ($x > 0$).

2.4.7 $f(x) = ax - x^3$, $f'(x) = a - 3x^2$.
Independent of the value of a, $x = 0$ is a fixed point. Since $f'(0) = a$, $x = 0$ is a stable fixed point for $a < 0$, but unstable for $a > 0$. For $a = 0$, $x = 0$ is also a stable fixed point, because then $f(x) = -x^3$ is positive for $x < 0$ and negative for $x > 0$.
If $a > 0$ there exist two other fixed points, $x = \pm\sqrt{a}$. Both are stable, because $f'(\pm\sqrt{a}) = -2a < 0$.
What happens at $a = 0$ is the prototype of a supercritical pitchfork bifurcation: one stable fixed point becoming unstable, with two stable fixed points branching off.

Strogatz, exercise 2.4.9

The equation $\dot{x} = -x^3$ may be written as

$$-\frac{\mathrm{d}x}{x^3} = \mathrm{d}t \ ,$$

assuming that $x \neq 0$. It may then be integrated directly:

$$\frac{1}{2x^2} = t - t_0 \ ,$$

where t_0 is an integration constant. For $t > t_0$ we have then the explicit solutions

$$x(t) = \pm\frac{1}{\sqrt{2(t - t_0)}} \ .$$

In addition we have the solution $x(t) = 0$.
For $t \to \infty$ we have that $x(t) \to 0$, but much more slowly than exponentially.

Strogatz, exercise 2.5.6

a) Water runs out of the bucket with velocity v through a hole of area a. The volume of water running through the hole during an infinitesimal time interval $\mathrm{d}t$ is $a|v|\,\mathrm{d}t$. The reduction of the water volume in the bucket, in the same time interval, is $A|\dot{h}|\,\mathrm{d}t$, where A is the surface area, h is the height of the water in the bucket, and $\dot{h} = \mathrm{d}h/\mathrm{d}t$.
Since the total volume of the water is constant, the two volumes must be equal:
$a|v|\,\mathrm{d}t = A|\dot{h}|\,\mathrm{d}t$, that is, $a|v| = A|\dot{h}|$.
Let us take the positiv direction to be up, then $v < 0$ and $\dot{h} < 0$, hence

$$av = A\dot{h} \ .$$

b) We imagine removing a layer of water of thickness $|\Delta h|$, from the top of the water. This water has a volume of $A\,|\Delta h|$ and a mass of $\rho A\,|\Delta h|$, where ρ is the density of water.
The kinetic energy of the same amount of water running out through the hole in the bottom, is $(1/2)(\rho A\,|\Delta h|)v^2$.
We now compare this kinetic energy to the reduction in the potential energy of the water. The net result is that water is removed at the top and emerges through the hole in the bottom, a distance h below the top. This reduces the total potential energy of the water in the bucket by $(\rho A\,|\Delta h|)gh$, where g is the acceleration of gravity.

Neglecting friction, and assuming that all the potential energy removed is converted into kinetic energy, we have that

$$(1/2)(\rho A\,|\Delta h|)v^2 = (\rho A\,|\Delta h|)gh \ ,$$

which gives the equation

$$v^2 = 2gh \ .$$

c) With $\dot{h} < 0$ we have, according to the two equations that

$$\dot{h} = \frac{a}{A}\,v = -\frac{a}{A}\,\sqrt{2gh} = -C\,\sqrt{h} \ ,$$

with $C = a\sqrt{2g}/A$.

d) We rewrite the equation as

$$-C\,\mathrm{d}t = \frac{\mathrm{d}h}{\sqrt{h}} \ ,$$

and integrate, with an arbitrary integration constant t_0. This gives that

$$-C(t-t_0) = 2\sqrt{h} \ .$$

Hence,

$$h(t) = \frac{C^2}{4}\,(t-t_0)^2 = \frac{ga^2}{2A^2}\,(t-t_0)^2 \ .$$

But the equation $\dot{h} = -C\,\sqrt{h}$, with $C > 0$, implies that $\dot{h} \leq 0$. Therefore we have to splice two different solutions to one solution:

$$h(t) = \begin{cases} \dfrac{ga^2}{2A^2}\,(t-t_0)^2 & \text{for} \quad t \leq t_0 \ , \\ \qquad 0 & \text{for} \quad t \geq t_0 \ . \end{cases}$$

This spliced solution solves the equation $\dot{h} = -C\,\sqrt{h}$ for all times. It has $h > 0$ before the time t_0, and $h = 0$ after t_0, and the time t_0 when there is no more water left is arbitrary and can not be determined from the equation.

Strogatz, exercise 2.6.1

The harmonic oscillator $m\ddot{x} = -kx$ is a two dimensional system in our terminology. We have to introduce a new variable, for example the momentum $p = m\dot{x}$, so that we get two first order equations: $\dot{x} = p/m$, $\dot{p} = -kx$. What we call the phase space of the oscillator, with variables x and p, is two dimensional.