TFY 4305 Nonlinear dynamics, autumn 2005.
Solutions to exercises

Strogatz, exercise 4.1.8

The equation $-\mathrm{d}V/\mathrm{d}\theta = f(\theta)$ has the solution

$$V(\theta) = -\int \mathrm{d}\theta\ f(\theta)\ .$$

The integration constant is uninteresting.
If $f(\theta) = \cos\theta$, then $V(\theta) = -\sin\theta$, which is a singlevalued function on the circle.
If $f(\theta) = 1$, then $V(\theta) = -\theta$, but this is not a singlevalued function on the circle.
The general condition for singlevaluedness is that

$$V(2\pi) - V(0) = -\int_0^{2\pi} \mathrm{d}\theta\ f(\theta) = 0\ .$$

Strogatz, exercise 5.1.9

c) The equation of motion is linear, of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

The matrix

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

has the characteristic equation, determining eigenvalues,

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 1\ ,$$

with roots $\lambda = \pm 1$. This shows that the origin is a saddle point, with one stable direction, which is the egenvector of eigenvalue -1, and one unstable direction, which is the egenvector of eigenvalue 1. The eigenvalue equation

$$A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

with eigenvalue $\lambda = -1$ gives the stable direction, which is the egenvector

$$\mathbf{u}_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

whereas the same eigenvalue equation with the eigenvalue $\lambda = 1$ gives the unstable direction, which is the egenvector

$$\mathbf{u}_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

The stable manifold of the fixed point at the origin is the special solution converging to the origin when $t \to \infty$,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C\mathrm{e}^{-t}\,\mathbf{u}_- = C\mathrm{e}^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

The solution is unique up to an integration constant C. Thus, the stable manifold is given by the equation $x = y$.

The unstable manifold is the special solution converging to the origin when $t \to -\infty$,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = D\mathrm{e}^{t}\,\mathbf{u}_+ = D\mathrm{e}^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

This solution is also unique up to an integration constant D. In other words, the unstable manifold is given by the equation $x = -y$.

d) For new variables $u = x + y$ and $v = x - y$ we get the equations

$$\dot{u} = \dot{x} + \dot{y} = -y - x = -u \;, \qquad \dot{v} = \dot{x} - \dot{y} = -y + x = v \;.$$

The solution with initial conditions $u(0) = u_0$ and $v(0) = v_0$ is $u(t) = u_0\mathrm{e}^{-t}$, $v(t) = v_0\mathrm{e}^{t}$.

e) The stable manifold is given by the equation $v = 0$, the unstable manifold has $u = 0$.

f) Since $u_0 = x_0 + y_0$ and $v_0 = x_0 - y_0$, we have the solutions

$$\begin{aligned} x(t) &= \frac{1}{2}\,(u(t) + v(t)) = \frac{1}{2}\,((x_0 + y_0)\mathrm{e}^{-t} + (x_0 - y_0)\mathrm{e}^{t}) = x_0 \cosh t - y_0 \sinh t \;, \\ y(t) &= \frac{1}{2}\,(u(t) - v(t)) = \frac{1}{2}\,((x_0 + y_0)\mathrm{e}^{-t} - (x_0 - y_0)\mathrm{e}^{t}) = y_0 \cosh t - x_0 \sinh t \;. \end{aligned}$$

We easily verify that this is a solution of the equations $\dot{x} = -y$, $\dot{y} = -x$ with initial values $x(0) = x_0$, $y(0) = y_0$. Since there exists only one solution, by the uniqueness theorem, this is the Solution, with a capital S.

Strogatz, exercise 5.1.13

The linear stability analysis at a saddle point (x_*, y_*) shows one stable direction $\mathbf{v}_-$ and one unstable direction $\mathbf{v}_+$, in fact this is the defining property of a saddle point. We may introduce these two directions as new coordinate axes, relative to this coordinate system an arbitrary point (x, y) has new coordinates (x_-, x_+) with

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_* \\ y_* \end{pmatrix} + x_-\mathbf{v}_- + x_+\mathbf{v}_+ \;.$$

The linearized equations of motion, expressed in the new coordinates, are

$$\dot{x}_- = \lambda_-\,x_- \;, \qquad \dot{x}_+ = \lambda_+\,x_+ \;,$$

where $\lambda_- < 0$ and $\lambda_+ > 0$ are the two eigenvalues characterizing the saddle point. The linearized equations of motion in this form form a gradient system, as defined in Chapter 7.2 in Strogatz,

$$\dot{x}_- = -\frac{\partial V}{\partial x_-} \,, \qquad \dot{x}_+ = -\frac{\partial V}{\partial x_+} \,,$$

with the potential function

$$V = V(x_-, x_+) = \frac{1}{2}\left(\lambda_- x_-^2 + \lambda_+ x_+^2\right) .$$

Since $\lambda_- < 0$ and $\lambda_+ > 0$, the graph of $V(x_-, x_+)$ looks like the saddle of a horse. Hence the name saddle point.

Strogatz, exercise 5.2.1

The equation of motion has the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

with

$$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} .$$

The characteristic equation is

$$0 = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 \,,$$

and it gives the eigenvalues $\lambda = \lambda_1 = 3$ and $\lambda = \lambda_2 = 2$. Two positive eigenvalues means that the origin is an unstable node.

The eigenvalue equation

$$A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

with $\lambda = \lambda_1 = 3$ or $\lambda = \lambda_2 = 2$ gives the two egenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

The general solution of the equation of motion is

$$\mathbf{x}(t) = c_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + c_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2 = \begin{pmatrix} c_1 \mathrm{e}^{3t} + c_2 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{3t} + 2c_2 \mathrm{e}^{2t} \end{pmatrix} .$$

Here c_1 and c_2 are constant coefficients, they may be determined for example from a given starting point at $t = 0$,

$$\mathbf{x}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{pmatrix} c_1 + c_2 \\ c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} .$$

This gives that

$$c_1 = 2x_0 - y_0 \ , \qquad c_2 = y_0 - x_0 \ ,$$

and,

$$\begin{aligned} \mathbf{x}(t) &= x_0 \begin{pmatrix} 2\mathrm{e}^{3t} - \mathrm{e}^{2t} \\ 2\mathrm{e}^{3t} - 2\mathrm{e}^{2t} \end{pmatrix} + y_0 \begin{pmatrix} -\mathrm{e}^{3t} + \mathrm{e}^{2t} \\ -\mathrm{e}^{3t} + 2\mathrm{e}^{2t} \end{pmatrix} \\ &= \begin{pmatrix} 2\mathrm{e}^{3t} - \mathrm{e}^{2t} & -\mathrm{e}^{3t} + \mathrm{e}^{2t} \\ 2\mathrm{e}^{3t} - 2\mathrm{e}^{2t} & -\mathrm{e}^{3t} + 2\mathrm{e}^{2t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} . \end{aligned}$$

This method is straightforward, but requires us to compute the egenvectors. Here is a method giving the answer without explicit knowledge of the eigenvectors. The equation

$$\dot{\mathbf{x}} = A\mathbf{x}$$

with A constant has the solution

$$\mathbf{x}(t) = \mathrm{e}^{tA}\,\mathbf{x}(0) \ ,$$

where the exponential function has the usual power series expansion,

$$\mathrm{e}^{tA} = I + tA + \frac{1}{2!}\,(tA)^2 + \cdots + \frac{1}{n!}\,(tA)^n + \cdots \ .$$

Thus, the problem is reduced to the summation of this series. Then we may use a mathematical theorem saying that every matrix A is a root of its own characteristic polynomial, which for our matrix A is $\lambda^2 - 5\lambda + 6$, so that A satisfies the equation $A^2 - 5A + 6I = 0$, where I is the unit matrix. From the equation

$$A^2 = 5A - 6I$$

follows that

$$\begin{aligned} A^3 &= AA^2 = A(5A - 6I) = 5A^2 - 6A = 5(5A - 6I) - 6A = 19A - 30I \ . \\ A^4 &= AA^3 = A(19A - 30I) = 19A^2 - 30A = 19(19A - 30I) - 30A = 331A - 570I \ . \end{aligned}$$

And so on. We might have used these relations directly to sum the series e^{tA}, but instead let us use a small trick. We realize that the answer must have the form

$$\mathrm{e}^{tA} = f(t)I + g(t)A \ ,$$

where $f(t)$ and $g(t)$ are two functions of t to be determined. We also realize that if λ is an eigenvalue, then $\lambda^2 = 5\lambda - 6$, and this relation gives that

$$\mathrm{e}^{t\lambda} = f(t) + g(t)\lambda \ ,$$

with the same two functions $f(t)$ and $g(t)$. Since the eigenvalues are $\lambda = 3$ and $\lambda = 2$, we have that

$$\begin{aligned} \mathrm{e}^{3t} &= f(t) + 3g(t) \ , \\ \mathrm{e}^{2t} &= f(t) + 2g(t) \ . \end{aligned}$$

Hence,

$$g(t) = e^{3t} - e^{2t} \; ,$$
$$f(t) = -2e^{3t} + 3e^{2t} \; .$$

And, finally,

$$e^{tA} = f(t)I + g(t)A = \begin{pmatrix} f(t) + 4g(t) & -g(t) \\ 2g(t) & f(t) + g(t) \end{pmatrix} = \begin{pmatrix} 2e^{3t} - e^{2t} & -e^{3t} + e^{2t} \\ 2e^{3t} - 2e^{2t} & -e^{3t} + 2e^{2t} \end{pmatrix} .$$

Strogatz, exercise 5.2.13

The equation of motion of the damped harmonic oscillator is

$$m\ddot{x} + b\dot{x} + kx = 0 \; .$$

Here x is the position, in one dimension, m is the mass, b the friction coefficient, and k is the spring constant. All these constants are assumed to be positive.

From classical mechanis we learn to introduce the momentum $p = m\dot{x}$, it gives the equations

$$\dot{x} = \frac{p}{m} \; , \qquad \dot{p} = -\frac{bp}{m} - kx \; .$$

An aside: does there exist a Hamiltonian function H such that

$$\dot{x} = \frac{\partial H}{\partial p} \; , \qquad \dot{p} = -\frac{\partial H}{\partial x} \; ?$$

If so, we must have that

$$0 = \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial p}\right) - \frac{\partial}{\partial p}\left(\frac{\partial H}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{p}{m}\right) - \frac{\partial}{\partial p}\left(\frac{bp}{m} + kx\right) = -\frac{b}{m} \; ,$$

hence $b = 0$, meaning there is no damping. Strangely enough, however, even when $b \neq 0$ it is possible to find a Hamiltonian function which is explicitly time dependent. Multiplying the original equation of motion by $e^{\alpha t}$, where $\alpha = b/m$, we may rewrite it like this:

$$\frac{d}{dt}\left(e^{\alpha t}\, m\dot{x}\right) + e^{\alpha t}\, kx = 0 \; .$$

Then we define $p = e^{\alpha t}\, m\dot{x}$ and

$$H = e^{-\alpha t}\,\frac{p^2}{2m} + e^{\alpha t}\,\frac{kx^2}{2} \; .$$

Back to our linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = A \begin{pmatrix} x \\ p \end{pmatrix} \qquad \text{with} \qquad A = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & -\frac{b}{m} \end{pmatrix} .$$

The sum of the eigenvalues of the matrix A is

$$\tau = \operatorname{Tr} A = -\frac{b}{m} < 0 \; ,$$

and their product is

$$\Delta = \det A = \frac{k}{m} > 0 \ .$$

Since $\Delta > 0$, both eigenvalues must have a negative real part, hence the origin is always a stable fixed point. It is a stable node if the eigenvalues are real, that is, if

$$\tau^2 - 4\Delta = \frac{b^2 - 4km}{m^2} \geq 0 \ , \qquad \text{or equivalently,} \qquad b \geq 2\sqrt{km} \ .$$

This is the case called overdamping: the damping is so large that the oscillator does not oscillate even once. The limiting case $b = 2\sqrt{km}$ is called critical damping.

In the opposite case, $b < 2\sqrt{km}$, the origin is a stable spiral, which means that the damping is small enough that the oscillator will really oscillate, with an exponentially decreasing amplitude.

The case of critical damping is interesting. Any matrix satisfies its own eigenvalue equation, and in particular, our 2×2 matrix A satisfies the equation

$$(A - \lambda_1 I)(A - \lambda_2 I) = 0 \ ,$$

where λ_1, λ_2 are the eigenvalues. When the damping is critical, it means that $\lambda_1 = \lambda_2$, and

$$(A - \lambda_1 I)^2 = 0 \ .$$

If a 2×2 matrix has two linearly independent eigenvectors with the same eigenvalue, then it is proportional to the identity matrix I. In the present case it is obvious that $A - \lambda_1 I \neq 0$, this means that A has two equal eigenvalues, but only one eigenvector. Since we have $(A - \lambda_1 I)^k = 0$ for $k \geq 2$, the series expansion for $\mathrm{e}^{t(A-\lambda_1 I)}$ gives that

$$\mathrm{e}^{tA} = \mathrm{e}^{\lambda_1 t}\, \mathrm{e}^{t(A-\lambda_1 I)} = \mathrm{e}^{\lambda_1 t} \left(I + t(A - \lambda_1 I) \right) \ .$$

Thus, the general solution of the equation of motion for the oscillator is

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \mathrm{e}^{tA} \begin{pmatrix} x(0) \\ p(0) \end{pmatrix} = \mathrm{e}^{\lambda_1 t} \left(I + t(A - \lambda_1 I) \right) \begin{pmatrix} x(0) \\ p(0) \end{pmatrix} .$$

There are now two possibilities. Let us define

$$\mathbf{w} = (A - \lambda_1 I) \begin{pmatrix} x(0) \\ p(0) \end{pmatrix} \ .$$

We may have $\mathbf{w} = 0$, this means that the starting point $(x(0), p(0))$ lies on the single eigenvector of A, and the solution is purely exponential,

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \mathrm{e}^{\lambda_1 t} \begin{pmatrix} x(0) \\ p(0) \end{pmatrix} .$$

The other possibility is that $\mathbf{w} \neq 0$. Then $\mathbf{w}$ is proportional to the single eigenvector of A, since the identity $(A - \lambda_1 I)^2 = 0$ implies that

$$(A - \lambda_1 I)\mathbf{w} = (A - \lambda_1 I)^2 \begin{pmatrix} x(0) \\ p(0) \end{pmatrix} = 0 \ .$$

Then as $t \to \infty$ we have that

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} \to t\,\mathrm{e}^{\lambda_1 t}\,\mathbf{w}\,.$$

We see that, in the case of critical damping, for an arbitrary staring point $(x(0), p(0))$ the system will converge exponentially towards the stable fixed point $(x, p) = (0, 0)$ along the direction of the single eigenvector of A.