Solutions, TFY4325 Astrofysikk, exercise 28/1 2005

Problem 1

In order to compute planetary orbits it is customary to make the assumption that both the Sun and the planets are point masses. That may sound reasonable, since the Sun and the planets are small compared to the distances between them. However, Newton found a better argument for assuming point masses: the Sun and planets are all almost perfectly spherically symmetric.
a) Show that the gravitational force between a spherically symmetric planet (or star) and a point mass outside it, is the same as if all the mass of the planet were at its centre.
b) Show that the gravitational force between two spherically symmetric planets is the same as if each of them had all its mass at its centre.

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a) Let us choose a coordinate system with origin at the centre of the planet, and with the point mass at the point $z = d$, $x = y = 0$. The force on the point mass we call $\vec{K}$, and it is clear, because of the rotational symmetry about the z axis, that $K_x = K_y = 0$. We have to compute K_z, and preferably get the result

$$K_z = -\frac{GMm}{d^2} \ ,$$

where G is Newton's gravitational constant, and M is the total mass of the planet.

We may do the computation in several ways. One way is the direct one. We assume that the mass density ρ is a function only of the distance from the origin, $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$, that is, $\rho = \rho(r)$. Then we have that

$$M = \int \mathrm{d}^3\vec{r}\,\rho(r) = 4\pi \int_0^\infty \mathrm{d}r\; r^2\,\rho(r) \ ,$$

and

$$K_z = -\int \frac{G\,(\rho(r)\;\mathrm{d}^3\vec{r})\,m}{(x^2+y^2+(d-z)^2)^{\frac{3}{2}}}\,(d-z) \ .$$

We introduce polar coordinates r, θ, φ, such that $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, and get that

$$K_z = -\int_0^\infty r^2\,\mathrm{d}r \int_{-1}^1 \mathrm{d}(\cos\theta) \int_0^{2\pi} \mathrm{d}\varphi\;\; \frac{G\rho(r)m(d-r\cos\theta)}{(r^2+d^2-2dr\cos\theta)^{\frac{3}{2}}} \ .$$

We simplify the notation by writing $u = \cos\theta$. Then

$$\begin{aligned}\frac{d-r\cos\theta}{(r^2+d^2-2dr\cos\theta)^{\frac{3}{2}}} &= \frac{d^2-r^2+r^2+d^2-2dru}{2d\,(r^2+d^2-2dru)^{\frac{3}{2}}} \\ &= \frac{1}{2d}\left(\frac{d^2-r^2}{(r^2+d^2-2dru)^{\frac{3}{2}}} + \frac{1}{(r^2+d^2-2dru)^{\frac{1}{2}}}\right) .\end{aligned}$$

Hence we get that

$$\begin{aligned}
K_z &= -\frac{2\pi Gm}{2d}\int_0^\infty r^2\,\mathrm{d}r\;\rho(r)\int_{-1}^1 \mathrm{d}u\left(\frac{d^2-r^2}{(r^2+d^2-2dru)^{\frac{3}{2}}}+\frac{1}{(r^2+d^2-2dru)^{\frac{1}{2}}}\right)\\
&= -\frac{\pi Gm}{d}\int_0^\infty r^2\,\mathrm{d}r\;\rho(r)\left(\frac{d^2-r^2}{dr\,(r^2+d^2-2dru)^{\frac{1}{2}}}-\frac{(r^2+d^2-2dru)^{\frac{1}{2}}}{dr}\right)\Bigg|_{u=-1}^{1}\\
&= -\frac{\pi Gm}{d^2}\int_0^\infty r\,\mathrm{d}r\;\rho(r)\left(\frac{d^2-r^2}{(r^2+d^2-2dr)^{\frac{1}{2}}}-\frac{d^2-r^2}{(r^2+d^2+2dr)^{\frac{1}{2}}}\right.\\
&\qquad\qquad\left.-\left(r^2+d^2-2dr\right)^{\frac{1}{2}}+\left(r^2+d^2+2dr\right)^{\frac{1}{2}}\right)\\
&= -\frac{\pi Gm}{d^2}\int_0^\infty r\,\mathrm{d}r\;\rho(r)\left(\frac{d^2-r^2}{|d-r|}-\frac{d^2-r^2}{d+r}-|d-r|+d+r\right)\\
&= -\frac{\pi Gm}{d^2}\int_0^\infty r\,\mathrm{d}r\;\rho(r)\left(\frac{d^2-r^2}{|d-r|}-(d-r)-\frac{|d-r|^2}{|d-r|}+d+r\right)\\
&= -\frac{\pi Gm}{d^2}\int_0^\infty r\,\mathrm{d}r\;\rho(r)\left(\frac{d^2-r^2-(d-r)^2}{|d-r|}+2r\right)\\
&= -\frac{\pi Gm}{d^2}\int_0^\infty r\,\mathrm{d}r\;\rho(r)\left(\frac{2r(d-r)}{|d-r|}+2r\right)\\
&= -\frac{4\pi Gm}{d^2}\int_0^d r^2\,\mathrm{d}r\;\rho(r)\\
&= -\frac{GmM}{d^2}\,.
\end{aligned}$$

The mass of the planet is

$$M = 4\pi\int_0^d r^2\,\mathrm{d}r\;\rho(r)\,,$$

when the distance d from the centre of the planet to the point mass m is larger than the radius of the planet, as we assumed.

Note that the above computation works even when the point mass is inside the planet. Then the result is that the gravitational force acting on the point mass is the same as if all the mass of the planet *closer to the centre than the point mass* were placed at the centre. The gravitational force from a spherically symmetric mass distribution outside the mass m is zero.

Another way, in fact a simpler way, to prove the same result is to compute the gravitational potential outside the planet, at a distance d from its centre, and show that it is

$$\phi = -\frac{GM}{d}\,.$$

In the same way as above we have that

$$\begin{aligned}
\phi &= -\int \frac{G\,(\rho(r)\ \mathrm{d}^3\vec{r})}{(x^2+y^2+(d-z)^2)^{\frac{1}{2}}} \\
&= -\int_0^\infty r^2\ \mathrm{d}r \int_{-1}^1 \mathrm{d}(\cos\theta) \int_0^{2\pi} \mathrm{d}\varphi\ \frac{G\rho(r)}{(r^2+d^2-2dr\cos\theta)^{\frac{1}{2}}} \\
&= -2\pi G \int_0^\infty r^2\ \mathrm{d}r\ \rho(r) \int_{-1}^1 \mathrm{d}u\ \frac{1}{(r^2+d^2-2dru)^{\frac{1}{2}}} \\
&= 2\pi G \int_0^\infty r^2\ \mathrm{d}r\ \rho(r)\ \frac{(r^2+d^2-2dru)^{\frac{1}{2}}}{dr}\Bigg|_{-1}^{1} \\
&= 2\pi G \int_0^\infty r^2\ \mathrm{d}r\ \rho(r) \left(\frac{|d-r|-(d+r)}{dr}\right) \\
&= -\frac{4\pi G}{d}\int_0^d r^2\ \mathrm{d}r\ \rho(r) - 4\pi G \int_d^\infty r\ \mathrm{d}r\ \rho(r) \\
&= -\frac{GM}{d}\ .
\end{aligned}$$

The last equality sign holds when $\rho(r) = 0$ for $r > d$.

A third way to compute is to use Gauss's law. When we define the gravitational field (also called the acceleration of gravity) as $\vec{g} = \vec{K}/m$, where $\vec{K}$ is the gravitational force on a small "test masse" m, then Gauss's law states that the flux of $\vec{g}$ through a closed surface is $4\pi G$ times the total mass inside the surface. Then we apply this theorem to a spherical surface of radius d about the centre of the planet, and assume that the gravitational field $\vec{g}$ is radial (this ought to follow from the rotational symmetry). This proof is of course good enough if we first accept Gauss's law. However, assuming Gauss's law to hold is more or less equivalent to assuming what we are asked to prove. In fact, the proof of Gauss's law is a computation much like the first computation above.

b) Knowing that the gravitational force between a spherically symmetric planet and a point mass outside it is as if all the mass of the planet were at its centre, we easily deduce that the gravitational force between two spherically symmetric planets is as if each of them had all its mass at its centre.

We use Newton's third law, that action equals reaction. It implies that the force from a point mass on a spherically symmetric planet is as if all the mass of the planet were at its centre. By assembling point masses into a spherically symmetric planet, we find, by exactly the same computation as above, that this spherically symmetric planet acts gravitationally on another spherically symmetric planet as if each planet had all its mass at its centre.

Problem 47 from Chapter 4, "Universe" 7th ed. (Problem 44 in 6th ed.)

Generalization of the problem from the text book:
Compute the tidal force from the Moon at an arbitrary point on the surface of the Earth.
Then we have to study the gravitational field from the Moon. If the centre of the Moon is at the position $\vec{r}_M$, then the acceleration of gravity due to the Moon, for a particle at the position $\vec{r}$, is

$$\vec{g} = \vec{g}(\vec{r}) = -\frac{GM_M}{|\vec{r} - \vec{r}_M|^3}\,(\vec{r} - \vec{r}_M)\ ,$$

where M_M is the mass of the Moon.
We may choose a suitable coordinate system, in which the centre of the Moon is at the origin, with coordinates $(x_M, y_M, z_M) = (0, 0, 0)$, and the centre of the Earth is at the point $(x_E, y_E, z_E) = (d, 0, 0)$, where d is the distance between the Earth and the Moon.
a) The whole Earth must have the same acceleration in the gravitational field from the Moon, otherwise it would get deformed. This acceleration equals the value of $\vec{g}(\vec{r})$ at the centre of the Earth. Why?
b) A small point mass m is located on the surface of the Earth, at the position
$(x, y, z) = (x_E + \Delta x, y_E + \Delta y, z_E + \Delta x) = (d + \Delta x, \Delta y, \Delta z)$. In order for this point mass to have the same acceleration as the Earth as a whole, the Earth has to push or pull on it with a small force $\vec{K} = K_x\vec{i} + K_y\vec{j} + K_z\vec{k}$. We assume that the displacements from the centre of the Earth, Δx, Δy and Δz, are sufficiently small that we may assume K_x, K_y and K_z to vary linearly with Δx, Δy and Δz. That is, we assume that

$$\begin{aligned} K_x &= K_{xx}\Delta x + K_{xy}\Delta y + K_{xz}\Delta z\ , \\ K_y &= K_{yx}\Delta x + K_{yy}\Delta y + K_{yz}\Delta z\ , \\ K_z &= K_{zx}\Delta x + K_{zy}\Delta y + K_{zz}\Delta z\ . \end{aligned}$$

We assume that $\vec{K} = 0$ for $\Delta x = \Delta y = \Delta z = 0$. Why is this assumption valid?
Compute the coefficients K_{xx}, K_{xy}, etc..
Why must we have for example that $K_{yy} = K_{zz}$?
Compute in particular $K_{xx} + K_{yy} + K_{zz}$.
Comment: The tidal force on the mass m is $-\vec{K}$, it is an example of what we call a fictitious force (because it has no reaction force, as Newton's 3rd law requires of a real force), and we experience it on the Earth because we are not in an inertial system, but rather in a reference system which is accelerated in the gravitational field from the Moon.
c) How does the tidal force vary with the distance d?
What consequence has this dependence on distance for the ratio between the tidal forces due to the Moon and the Sun?

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a) We assume, as a good approximation, that the Earth and Moon are both spherically symmetric, so that they act gravitationally as if they were both point masses. The acceleration of the Earth as a whole is therefore equal to the acceleration of gravity caused by the Moon, computed at the point which is the centre of the Earth. With the Moon at the origin, $(x_M, y_M, z_M) = (0, 0, 0)$, and the Earth at $(x_E, y_E, z_E) = (d, 0, 0)$, the Earth has an acceleration $\vec{g}_0$ with components $g_{0y} = g_{0z} = 0$,

$$g_{0x} = -\frac{GM_M M_E}{d^2}\ .$$

We choose to disregard the fact that the Earth rotates about its own axis. Then the whole Earth must have the same acceleration $\vec{g}_0$.

b) A small point mass m is located on the surface of the Earth (or above or below the surface) at the position $\vec{r} = \vec{r}_E + \Delta\vec{r}$. For this mass to have the same acceleration $\vec{g}_0$ as the whole Earth, the Earth has to push or pull on it with a small force $\vec{K}$, so that

$$\vec{K} + m\vec{g}(\vec{r}) = m\vec{g}_0 \ .$$

The left hand side of this equation is the sum of the forces acting on the mass m, and the right hand side is the acceleration, as seen from an inertial system.

Now we choose instead to regard the same physical system from an accelerated reference system in which the Earth is at rest. In this system it looks as if we have to use the force $\vec{K}$ to keep the mass m at rest, in other words, it looks as if the mass m is subject to a force $-\vec{K}$, which is a so called fictitious force, present because we describe the phenomena in an accelerated reference system. This fictitious force,

$$\vec{K}_T = -\vec{K} = m(\vec{g}(\vec{r}) - \vec{g}_0) \ ,$$

is the tidal force.

Thus, at the centre of the Earth we have $\vec{K}_T = 0$. If the distance $\Delta\vec{r}$ from the centre of the Earth is small enough, we may expand in a series to first order only in $\Delta\vec{r}$. The expansion gives in particular for the x component of the tidal force,

$$K_{Tx} = m(g_x(\vec{r}_E + \Delta\vec{r}) - g_x(\vec{r}_E)) = m\left(\left.\frac{\partial g_x}{\partial x}\right|_{\vec{r}=\vec{r}_E} \Delta x + \left.\frac{\partial g_x}{\partial y}\right|_{\vec{r}=\vec{r}_E} \Delta y + \left.\frac{\partial g_x}{\partial z}\right|_{\vec{r}=\vec{r}_E} \Delta z \right) .$$

Since

$$g_x = -\frac{GM_M x}{r^3} \ ,$$

with $r = \sqrt{x^2+y^2+z^2}$, and

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r} \ , \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \ , \qquad \frac{\partial r}{\partial z} = \frac{z}{r} \ ,$$

we have that

$$\frac{\partial g_x}{\partial x} = -\frac{GM_M}{r^3} + \frac{3GM_M x^2}{r^5} \ , \qquad \frac{\partial g_x}{\partial y} = \frac{3GM_M xy}{r^5} \ , \qquad \frac{\partial g_x}{\partial z} = \frac{3GM_M xz}{r^5} \ ,$$

and hence, in our linear approximation,

$$K_{Tx} = \frac{2GM_M m}{d^3}\Delta x \ .$$

Since

$$g_y = -\frac{GM_M y}{r^3} \ ,$$

we have similarly that

$$\frac{\partial g_y}{\partial x} = \frac{3GM_M xy}{r^5} \ , \qquad \frac{\partial g_y}{\partial y} = -\frac{GM_M}{r^3} + \frac{3GM_M xy}{r^5} \ , \qquad \frac{\partial g_y}{\partial z} = \frac{3GM_M y^z}{r^5} \ ,$$

and consequently, in the same linear approximation,

$$K_{Ty} = -\frac{GM_M m}{d^3}\Delta y \ .$$

Thanks to the rotational symmetry about the x axis we may say immediately that

$$K_{Tz} = -\frac{GM_M m}{d^3}\,\Delta z\ .$$

We see that if we write

$$\begin{aligned} K_{Tx} &= K_{Txx}\Delta x + K_{Txy}\Delta y + K_{Txz}\Delta z\ , \\ K_{Ty} &= K_{Tyx}\Delta x + K_{Tyy}\Delta y + K_{Tyz}\Delta z\ , \\ K_{Tz} &= K_{Tzx}\Delta x + K_{Tzy}\Delta y + K_{Tzz}\Delta z\ , \end{aligned}$$

we have that

$$K_{Txx} = \frac{2GM_M m}{d^3}\ , \qquad K_{Tyy} = K_{Tzz} = -\frac{GM_M m}{d^3}\ ,$$

and all the other coefficients vanish. Note that

$$K_{Txx} + K_{Tyy} + K_{Tzz} = 0\ .$$

This is simply the field equation for the gravitational field from the Moon, which states that the divergence vanishes, $\nabla \cdot \vec{g} = 0$.

c) The tidal force varies with the distance d as $1/d^3$. This behaviour has an interesting consequence for the ratio between the tidal forces due to the Moon and the Sun. As is well known, the ratio of the distances to the the Sun and the Moon, d_S and d_M, is very nearly the same as the ratio of the radii of the Sun and the Moon, R_S and R_M:

$$\frac{d_S}{d_M} = \frac{R_S}{R_M}\ .$$

We know this because we see total eclipses of the Sun by the Moon where the face of the Moon covers exactly the face of the Sun. According to our formulae above, the ratio of the tidal forces here on Earth due to the Moon and the Sun is

$$\frac{M_M d_S^3}{M_S d_M^3} = \frac{M_M R_S^3}{M_S R_M^3} = \frac{M_M \mathcal{V}_S}{M_S \mathcal{V}_M} = \frac{\rho_M}{\rho_S}\ ,$$

where $\mathcal{V}_S$ and $\mathcal{V}_M$ are the volumes, and ρ_M and ρ_S are the average densities.

Hence, when the Moon causes tides that are three times as large as the tides caused by the Sun, it is because the Moon has three times as high average density as the Sun.